Thoughts on Algebraic Topology

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1 What's this about?

This is going to be a rapid introduction/motivation for a number of topics in Algebraic Topology, that took me a while to understand in my own way. I shall assume that the reader is not ignorant about the terms discussed, and I shall keep notation to a minimum. The first few sections shall deal with the fundamental group and covering spaces, the next few will be about homology (simplicial and singular), chain complexes and chain homotopy. Rigour has been sacrificed.

2 Fundamental group

Let us say our goal is to distinguish and classify spaces upto homeomorphism. For example, we would like to say that the different $\mathbb{R}^n$s and S^ns are not homeomorphic to each other, i.e., they are different things, but we would also like to answer, quantitatively, how different these objects are.

The first obvious thing is to look for properties that remain the same under homeomorphisms, properties like compactness, connectedness, path connectedness, and their local versions. These are called topological properties. However, there aren't enough topological properties to classify and distinguish spaces. All S^ns are compact, path connected, but they are different.

There is something we can do with connectedness, namely cut points. Removing a point in $\mathbb{R}$ disconnects it, but doesn't disconnect the other $\mathbb{R}^n$s. Removing two points from S^1 disconnects it into two components, but the same doesn't happen in S^2 (this can be seen with stereographic projection, for example). Here's another way to look at cut points. Consider $0 \in \mathbb{R}$. It being a cut point tells us that every path from 1 to -1 (for $\mathbb{R}$ is path connected) must pass through 0. Similarly, $1, -1$ form a cut set in S^1, and it tells us that every path from i to $-i$, for example, must pass through one of these points.

There may be other ways to look at cut points and cut sets, but for the moment, let us consider what we have above. Essentially, it boils down to how many paths there are between two given points. But the number of paths is enormous given that we can always reparametrize it, so we must cut down this number by making certain identifications.

So, we identify two paths if we can deform one to the other, this is homotopy.

Now, instead of two points, let us focus on just loops at a point. You can think of loops as paths with the same end points. So, we ask about the number of loops at a point. One benefit of loops is that we can compose, or take the product of two loops. This is the path product and is not available for arbitrary paths and this notion of a product gives us more structure than just a number. With this notion of a product, we get the fundamental group.

Another way to look at the fundamental groups of a space X is to think of it as all maps from S^1 to X identified upto homotopy. Note that we must make an arbitrary, artificial base point for all our loops. For a path connected space, this doesn't matter in that it doesn't change the group structure. However, it still matters in some ways because changing the base point sure gives us the same groups, but there is no natural isomorphism between the groups. Let us say we have two base points x_0, x_1 and the associated fundamental groups G_0, G_1. The isomorphism between these two groups is in a correspondence with the paths from x_0 to x_1 identified upto homotopy. We shall come to this later. For now, we move onto covering spaces.

3 Covering spaces

Suppose we have a continuous map $f\colon X \to Y$ where X, Y are path connected spaces. Then f induces a group homomorphism $f_*\colon \pi(X; x_0) \to \pi(Y; y_0)$ where $f(x_0) = y_0$.

For f_* to be surjective, given a loop $l\colon [0,1] \to Y$ in Y based at y_0, we must lift it to X, i.e., obtain a loop $\tilde{l}$ based at x_0 such that $f \circ \tilde{l} = l$. The best way to do this is to force f to have local inverses, i.e., around each $y \in Y$ there should be a neighbourhood V and a continuous function $g\colon V \to X$ under which V is homeomorphic to its image with inverse f. Allowing such a V would enable us to lift $l([0, \epsilon))$ for some $\epsilon > 0$ to X, however there is no guarantee that the base point y_0 lifts to x_0 and base points are fixed beforehand.

Therefore, we instead force f to satsify the following property. Around each $y \in Y$ and $x \in f^{-1}(y)$ there are neighbourhoods $x \in U \subseteq X, y \in V \subseteq Y$ such that $f|_U\colon U \to V$ is a homeomorphism. So, given a the loop l, we first take a neighbourhood V_0 around $l(0) = y_0$ and lift l to some $\tilde{l}$ on the interval $[0, \epsilon_1]$ for some $\epsilon_1 > 0$. Next, we take a neighbourhood V_1 around $l(\epsilon_1)$ and extend the lift $\tilde{l}$ to some $[0, \epsilon_2]$ by using neighbourhoods around $l(\epsilon_1)$ and $\tilde{l}(\epsilon_1)$.

It is easy to see how this process continues, except it is not clear whether it terminates. We can remedy this if around each $y \in Y$, the neighbourhood V is independent of the choice of $x \in f^{-1}(y)$. This way, the ϵ_2 is fixed regardless of how the previous lift $\tilde{l}$ behaves. So, we force f to satisfy the following. To have around each $y \in V$ a neighbourhood V such that for each $x \in f^{-1}(y)$ there's a neighbourhood U_x which maps to V homeomorphically under f.

Then we can cover l with open neighbourhoods $\{V_y\}_{y \in l}$ that are as described as above and by Lebesgue covering number, obtain a subdivision $0 = s_0 < s_1 < \cdots < s_n = 1$ such that $l([s_i, s_{i+1}])$ is contained in some V_y. First set $\tilde{l}(0) = x_0$. Suppose $\tilde{l}$ is defined on $[0, s_i]$.

The segment $l([s_i, s_{i+1}])$ lies in some V_y, and for every $x \in f^{-1}(y)$, there's a U_x mapping homeomorphically to y. The element $\tilde{l}(s_i)$ must be in at least one U_x, pick one arbitrarily and extend $\tilde{l}$ to $[0, s_{i+1}]$. Thus, we can lift l to a continuous $\tilde{l}$ satisfying $f \circ \tilde{l} = l$.

However, we made a choice of which U_x to choose and this means that there may be non unique liftings of l. We can avoid this if we force the $U_x, x \in f^{-1}(y)$ to be disjoint. This pins down the definition of a covering map. It turns out that not only do covering maps allow lifting of paths and loops, they also lift homotopies; the proof is similar and involves subdividing I^2.

However, this lifting property doesn't always send loops based at y_0 to loops based at x_0, it lifts any loop l to a path $\tilde{l}\colon [0, 1] \to X$ with $\tilde{l}(0) = x_0$ and $\tilde{l}(1)$ being some point in the fibre of y_0, in other words, the lifting property provides a map $\phi\colon \pi(Y; y_0) \to f^{-1}(y_0)$ (one has to prove that homotopic loops have homotopic lifts and the disjointedness, which implies uniqueness of lifts, gurantees that the end points are the same). Because X is path connected, this map is surjective and when X is simply connected, it is bijective.

So when f is a convering map, f_* need not be surjective, but the lifting correspondence can give us some information about the $\pi(Y; y_0)$. However, since we can lift homotopies (uniquely), the map f_* is injective.

4 Complexes and Homology

Previously we talked about the number of paths between two points. We first considered loops, which can be thought of as paths between a point and itself.

Now we consider two points fixed before hand. Unlike the case of loops, we do not have a path product here. Again, we identify paths upto homotopy. What are these paths? Well, they would be all maps from $[0, 1]$ to X identified upto homotopy.

Maps from $[0, 1]$ to X are paths between 2 points, so a natural extension is to consider paths between n points, so we look at maps from graphs to X. We can further generalise this by looking at maps from solid triangles, tetrahedrons and so on, i.e., maps from the simplexes.

Now, even though there is no group structure as before, we can form a free group over $\mathbb{Z}$ by looking at all these maps from $[0, 1]$, and triangles, and tetrahedrons and so on. Now, the difference between paths and loops is that the end points are the same in loops, the end points are maps from a single point to X, and $l\colon [0, 1] \to X$ is a loop if $l(1) = l(0)$, or $l(1) - l(0) = 0$ in the free group $\mathbb{Z}\{\text{maps}\colon * \to X\}$. Another reason to think of the free group: a square is the union of two triangles, so a map from the unit square to X (i.e., a homotopy between two paths) can be considered as a sum of two maps from the triangle.

The signs are related to orientation on our maps from simplexes. Simplexes in $\mathbb{R}^n$ come with two orientations and given one such orientation, we can orient the boundary faces. And, just as in the case of paths, we get a boundary homomorphism. From here we get a chain complex and we have the associated homology groups. However, what maps do we consider is still a question, do we consider all possible maps from an n-simplex to X? If so, the homology group is called the singular homology groups.

There is the following notion of a reduced homology. To the usual singular chain complex, we add $\mathbb{Z}$ and a map $\epsilon : C_0(X) \to \mathbb{Z}$ such that $\epsilon(\sum n_i \sigma_i) = \sum n_i$. Following [1], we denote the corresponding groups by $\tilde{H}_n(X)$. Through out, some care has to be taken when dealing with the zeroeth homology groups, we will however not bother about that in this hand-wavy exposition. Whatever results are stated for the regular homology groups (we shall essentially be dealing only with the equality of groups computed in different ways) will hold for the reduced homology groups as well.

However, things are greatly simplified if we restrict the class of spaces we look at. Simplicial complexes are spaces that are given a simplicial structure. Simplicial homology comes from looking at the chain complex of free groups generated by the n-simplexes in a simplicial complex X.

A Δ-complex is generated by inductively attaching higher dimensional simplexes, except that the boundary must also agree with previous attachments while CW complexes don't have such a restriction. And in each of those we have a homology, and all of these agree with the singular homology whenever defined.

5 Chain homotopy

Finally we come to the concept of chain homotopy. Let us consider the singular homology groups because they are much more general and theoretically simpler than the others. Let X, Y be two topological spaces and let $C., D.$ be the associated chain complexes.

If $f\colon X \to Y$ is a continuous map, then it induces a map between the chain complexes and the associated homology groups. Suppose $f, g\colon X \to Y$ are homotopic maps, then given an n-simplex $\sigma\colon \Delta^n \to X$, we have to homotopic maps $f \circ \sigma, g \circ \sigma$. These being homotopic means that there is a homotopy $H\colon \Delta^n \times [0,1] \to Y$.

Now, $\Delta^n \times [0,1]$ is a sum of $n+1$ simplexes, and so H is an element of D_{n+1}. We look at what happens to the boundaries. We have two elements $f \circ \sigma, g \circ \sigma \in D_n$, and by restricting H to $\partial\Delta^n \times [0,1]$, we have an element of D_n, and we also have ∂H coming from D_{n+1} to D_n. Intuitively, if H is a solid "prism", then ∂H is it's boundaries, which include a top face which maps to $f \circ \sigma$ and a bottom face which maps to $-g \circ \sigma$ (because of an outward pointing normal, say) and the sides, which are the restriction of H to $\partial\Delta^n \times [0,1]$. Therefore, if we denote the map sending each σ (in any dimension) to the correspoding homotopy between $f \circ \sigma, g \circ \sigma$ by P (extending P in a linear fashion to the free group), then we have

$$\partial P + P\partial = f_* - g_*.$$

The sides of the prism cancel out and we are left with the difference of the top and bottom faces. We take this to be a starting point for introducing the concept of chain homotopies. Given chain complexes $C., D.$ and maps $f, g\colon C., D.$, we say f, g are homotopic if there's a map P of degree 1, i.e., $P_n\colon C_n \to D_{n+1}$ such that $Pd_C + d_D P = f - g$.

Note to the reader: For an actual proof, see [1].

To every chain complex there is an associated sequence of homology groups, which measure how far the complex is from being exact at any given point in the complex. This is the homology functor, and it is quite easy to prove that homotopic chain maps induce the same maps between the homology groups.

Furthermore, we now have the category of chain complexes, whose objects are chain complexes and morphisms are chain maps. With a little bit of effort, one can show that we also have a category whose objects are chain complexes and morphisms are maps between them identified up to a chain homotopy. We can then say when two chain complexes are homotopic to each other, and in this case, their homology groups are going to be isomorphic. This is a great tool to compute homology groups of topological spaces.

6 Reduced homology, Excision, Mayer-Vietoris

It is often difficult to directly compute homology groups of a space. One route to simplification is to consider a different homotopic space. Another thing one can do is to try to quotient out parts of the space. Suppose X is a space and A is a subspace and suppose we know the homology groups of A and X/A, then one would like to somehow patch the two to get the homology groups of X. This is where the concept of reduced homology comes in.

We form the quotient group $C_n(X, A) = C_n(X)/C_n(A)$ by killing all the simplices landing in A. Since the boundary of a simplex in A also lies in A, we have the corresponding relative chain complex and the associated relative homology groups. Moreover, we have the exact sequence

$$0 \longrightarrow C_n(A) \longrightarrow C_n(X) \longrightarrow C_n(X, A) \longrightarrow 0$$

Here, the first map is the inclusion, and the second is the quotient map. and both commute with the boundary maps. Whenever we have such an exact sequence of chain complexes, we have an long exact sequence between the homology groups:

$$\ldots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow \ldots$$

So, if we know the homology groups of A and the relative homology groups, then we may be able to solve for the homology groups of X. There is yet another tool and this is the excision theorem, as a corollary of which, we have $H_n(X, A) = H_n(X/A)$ for nice enough pairs (X, A). It so happens that for a point $x_0, H_n(X, x_0) = \tilde{H}_n(X)$.

Furthermore, if $f, g\colon (X, A) \to (Y, B)$ are maps between pairs of spaces, then there are induced maps f_*, g_* between the relative homology groups and, as before, when f, g are homotopic, the induced homomorphisms are equal.

6.1 Excision

So, we have a topological space and we have the idea of homology, where we look at some maps from simplices (which are quite simple objects) to the space and an associated free group and boundary maps which then give us a homology. Singular homology applies to all spaces, but given a delta complex structure or a CW complex structure on a space, we have an associated homology too. The point is in showing that the corresponding homology groups are equal. There is no reason to expect them to be equal, but by constructing a few clever chain homotopy maps, one can show that the associated complexes are homotopic, which would mean that the homology groups are equal.

Suppose $\mathcal{U}$ is an open cover of X, then we can form a chain complex, $C_n^{\mathcal{U}}(X)$ of simplices landing in elements of $\mathcal{U}$ and we get an associated homology. Barycentric subdivision is a fantastic tool to use at this point. Basically, given a simplex, we can subdivide it into smaller (smaller in a real sense, the diameter decreases) simplices. As in [1], we do this in four steps.

First, given a simplex Δ in regular Euclidean space, one can subdivide it and write it as a union of simplices whose diameters are at most $n/n + 1$ times the diameter of Δ where n is the dimension of Δ (a question of well definedness does arise here, but all we need is that it's possible to describe points of Δ with n coordinates).

Next, we go to linear chains. Let Y be a convex subset of some Euclidean space, let $LC_n(Y)$ denote all linear (or simplicial, meaning that we only need to know where the vertices go) maps from n simplices to Y (which is convex). This is a chain complex and we extend this by appending $\mathbb{Z}$ as $LC_{-1}(Y)$ (and sending generators of $LC_0(Y)$ to 1) for the sake of convenience. Given a point $b \in Y$, we have a map $b\colon LC_n(Y) \to LC_{n+1}(Y)$ obtained by basically taking b to be a cone point over any simplex (and extended linearly to $LC_n(Y)$). It is easy to show that $\partial b + b\partial = 1$, i.e., b is a chain homotopy between 0 and identity. Let $S\colon LC_n(Y) \to LC_n(Y)$ denote the barycentric subdivision operation defined inductively by being identity at the $n = 0, -1$ levels and $S\lambda = b_\lambda(S\partial\lambda)$ for any simplex λ (and extended linearly) where b_λ is the geometric barycentre of λ. Intuitively, we take the boundary, subdivide it, and then attach b_λ as the cone point. At $n = 0, -1$ levels, we have $S\partial = \partial S$, and at higher levels

$$\begin{aligned}
\partial S\lambda &= \partial b_\lambda(S\partial\lambda) \\
&= S\partial\lambda - b_\lambda(\partial^2 S\lambda) \text{ by } \partial b + b\partial = 1 \text{ and induction} \\
&= S\partial\lambda
\end{aligned}$$

So, having constructed S, we have to show that it is actually chain homotopic to identity, i.e., we need to find a $T\colon LC_n(Y) \to LC_{n+1}(Y)$ such that $\partial T + T\partial = 1 - S$. Again we do this inductively. At the $n = -1$ level, since $S = 1$, we set $T = 0$. For higher n, we set $T = b_\lambda(\lambda - T\partial\lambda)$ on the generators and extend linearly. We verify

$$\begin{aligned}
\partial T\lambda &= \partial b_\lambda(\lambda - T\partial\lambda) \\
&= \lambda - T\partial\lambda - b_\lambda\partial\lambda + b_\lambda\partial T\partial\lambda[b\partial + \partial b = 1] \\
&= \lambda - T\partial\lambda - b_\lambda[S\partial\lambda + T\partial^2\lambda] \text{ by induction} \\
&= \lambda - T\partial\lambda - S\lambda
\end{aligned}$$

So, T is a chain homotopy between $1, S$.

Next we extend this subdivision operation to general singular chains. The idea is that we first subdivide the simplex as a subset of the Euclidean space and then map it to X. Since the subdivision of a simplex continues to lie inside the simplex, this works fine. The same works for T as well. And the identities and chain homotopies continue to hold.

Lastly, we define iterated subdivisions in a similar manner. Since $1, S$ are chain homotopic, so are $1, S^m$ for any m. Explicitly, a chain homotopy is given by $D_m = \sum_{i=0}^m TS^i$. Now, let $\sigma\colon \Delta^n \to X$ be an n-simplex. Since it's image is compact, and upon taking subdivisions, the diameter decreases, there is some m such that $S^m(\sigma)$ lies in $C^{\mathcal{U}}(X)$. However, this m will depend on σ. Technically, we have $S^m\sigma \in C_n(X)$, but each of it's constituent simplices lies in some element of $\mathcal{U}$. So, $S^m\sigma$ is in the image of the inclusion $C_n^{\mathcal{U}}(X) \to C_n(X)$. It is in this way that the mth subdivision lies in $C^{\mathcal{U}}(X)$.

We want to show that the chain complexes $C(X), C^{\mathcal{U}}(X)$ are homotopic. We already have the inclusion map $\iota\colon C^{\mathcal{U}}(X) \to C(X)$. Subdivision is an immediate candidate for the homotopy inverse, but we must take a different m for each σ. As for the chain homotopy, consider $D\colon C_n(X) \to C_{n+1}(X)$ sending $\sigma \mapsto D_{m(\sigma)}\sigma$ on the generators and then extended linearly.

It is not clear how to implement subdivision as the homotopic inverse of ι. What we do is that we construct a $\rho\colon C_n(X) \to C_n(X)$ with image in $C_n^{\mathcal{U}}(X)$, such that $\partial D + D\partial = 1 - \rho$. Since we have defined D, we take this equation to be the definition of ρ, then ρ is a chain map homotopic to identity.

Pick a generator σ, then

$$\begin{aligned}
\rho(\sigma) &= \sigma - \partial D\sigma - D(\partial\sigma) \\
&= \sigma - \partial D_{m(\sigma)}\sigma - D(\partial\sigma) \\
&= S^{m(\sigma)}\sigma + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \text{ since } \partial D_m + D_m\partial = 1 - S^m
\end{aligned}$$

The first term is in $C_n^{\mathcal{U}}(X)$ by definition of $m(\sigma)$. The remaining terms are linear combinations of $D_{m(\sigma)}(\sigma_j) - D_{m(\sigma_j)}(\sigma_j)$ where σ_j is the restriction of σ to the jth face of Δ^n.

It is clear that $m(\sigma_j) \leq m(\sigma)$, so the difference above, by definition of D_m will consist of terms of the form $TS^i(\sigma_j)$ for $i \geq m(\sigma_j)$, hence is entirely in $C_n^{\mathcal{U}}(X)$ (T sends $C_{n-1}^{\mathcal{U}}$ to $C_n^{\mathcal{U}}$ as everything happens inside the image of σ).

So, ρ can be veiwed as a map $C_n(X) \to C_n^{\mathcal{U}}(X)$ and we have $1-\iota\rho = \partial D + D\partial$ and conversely, since D is zero on $C_n^{\mathcal{U}}(X)$, we see that $\rho\iota = 1$ on $C^{\mathcal{U}}$. Note that for $\sigma \in C_n^{U}(X)$, we have $m(\sigma) = 0$, hence $D = 0$.

Thus, the two chain complexes are homotopic, and we have an easier way to compute the singular homology groups of X. Note in particular that the isomorphism is induced by the inclusion map.

Now, if $\mathcal{U} = \{A, B\}$, then this result gives us an isomorphism $H_n(X, A) \cong H_n(B, A \cap B)$ via the inclusion. One should verify that the maps above factor through the quotients, see [1].

A pair (X, A) is said to be a good pair if A is a closed subset which is the deformation retract of an open subset. Let an open set V deformation retract onto A and let q denote the quotient map $q\colon X \to X/A$.We get the following commutative diagram

$$\begin{array}{ccccc}
H_n(X,A) & \longrightarrow & H_n(X,V) & \longleftarrow & H_n(X-A, V-A) \\
{\scriptstyle q^*}\big\downarrow & & \big\downarrow{\scriptstyle q^*} & & \big\downarrow{\scriptstyle q^*} \\
H_n(X/A, A/A) & \longrightarrow & H_n(X/A, V/A) & \longleftarrow & H_n(X/A - A/A, V/A - A/A)
\end{array}$$

where the upper left and bottom right horizontal maps are isomorphism by the following observation : we have the triple (X, V, A) and an exact sequence (induced by inclusions and quotients, hence they commute with the boundary homomorphisms)

$$0 \longrightarrow C_n(V, A) \longrightarrow C_n(X, A) \longrightarrow C_n(X, V) \longrightarrow 0$$

Since V is deformation retracts onto A, using the long exact sequence for (V, A) we are forced to have $H_n(V, A) = 0$, and using this and the long exact sequence associated to the exact sequence of complexes above, we conclude that $H_n(X, V) \cong H_n(X, A)$.

In a similar manner, the lower left horizontal map is an isomorphism for the deformation retraction of V to A induces one from V/A to A/A. The other two horizontal maps are isomorphisms by the excision theorem.

Finally, the rightmost vertical map is an isomorphism because the quotient map is a homeomorphism when restricted to $X - A$. By the commutativity of the diagram, the left vertical map is an isomorphism as required. One needs to be a bit careful in veryfing that the diagram is indeed commutative, however things work out because they work out at the level of chain complexes (the boundary map is well behaved in these regards).

6.2 Other equivalences

Using the excision theorem, one is able to prove a number of results. However, one can do better. While we have reduced our calculations to $C^{\mathcal{U}}(X)$, we are still dealing with arbitrary simplices. Most ordinary spaces are either themselves simplicial complexes (spheres, triangulated spaces, polyhedra etc.) or can be given a CW-complex structure. In such cases, we have an associated homology groups.

We first form the free groups taking the n-dimensional simplices or cells as a basis, and we have the obvious boundary map, so we obtain a chain complex, and thus a homology. One can show that the associated homologies are the same, see [1]. In the case of finitely many cells, the nth homology group has a finite rank (the Betti number) and some torsion parts because it is a finitely generated abelian group.

6.3 Mayer-Vietoris theorem

Suppose $A, B \subset X$ and X is the union of their interiors. We then have a short exact sequence

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \longrightarrow 0$$

where ϕ sends x to $(x, -x)$ (via the inclusion of the intersection in A, B) and ψ is addition. Here $C_n(A + B)$ denotes those n-chains that are sums of chains in A, chains in B.

From this short exact sequence we have a long exact sequence

$$\ldots \longrightarrow H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} \ldots$$

where we are using the excision theorem.

7 Homotopy groups

Earlier we defined the fundamental group to be the homotopy classes of all maps from S^1 to X. Intuitively, loops enclose 1 dimensional holes, so to capture higher dimensional holes, we need to look at the homotopy classes of maps from spheres to X. Using the suspension operation, we can make this collection a group, the homotopy groups.

The path product is essentially obtained by travelling around the circle twice, say f, g are loops that fix one (we can homotope f, g to be as such using rotations), then the path productis obtained by identifying $1, -1$ to obtain a wedge of two circles, and then map one circle via f and the other via g. A generalization to higher dimensions is to take the wedge. Consider S^n. To obtain the product of $f, g \colon S^n \to X$, we can surely consider a map $S^n \wedge S^n \to X$ sending the first sphere via f and the other via g. But we need to obtain a map $S^n \to X$, and to do this, we would need a map $S^n \to S^n \wedge S^n$.

Moreover, such a map should be nice enough in that projecting to each of the spheres should result in identity (if we were to take the product of f with a constant map, for example). Well, since we are working up to homotopy, it suffices for the composition to be homotopic to identiy. Such considerations lead to the concept of cogroups (these are groups, but with arrows reversed), and H-cogroups, i.e., arrows are to be identified up to homotopy.

The nice thing is that the suspension of a space is a H-cogroup and S^n is a suspension of S^{n-1} and we know that S^1 is a cogroup. Thus, we have the higher homotopy groups, see [2].

Lastly, we considered points, then paths between points identified up to homotopy. This construction can be repeated. We can consider homotopies between homotopies ad infinitum, and this results in some nice structure that I am unaware of at the time of writing.

8 Winding number and degree

Let γ be a curve in the punctured complex plane, then we define the winding number of γ around 0 to be

$$w(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} dz.$$

One can show that the winding number is an integer and by Cauchy's theorem (which one?) it depends only on the homotopy class of γ. So, it suffices to consider winding numbers of elements of $\pi_1(S^1)$ since the punctured plane deforms to the unit circle. By additivity of integrals, the winding number of $\gamma_1 \cdot \gamma_2$ (the path product) is the sum of their individual winding numbers, thus we have a group homomorphism $w \colon \pi_1(S^1) \to \mathbb{Z}$ sending 1 to 1 because the identity loop has a winding number of 1.

Given an element of $\gamma \in \pi_1(S^1)$, we have the degree of the map $\gamma_* \colon H_1(S^1) \to H_1(S^1)$. Again, this degree depends only on the homotopy class, so we have a map $\deg \colon \pi_1(S^1) \to \mathbb{Z}$. Our aim is to show that the two maps are the same. Since $\deg(1) = 1$, it suffices to show that the degree is a group homomorphism, whence it follows that the winding number of a curve is the same as the degree of the induced maps on the homology group.

Now, let f, g be loops fixing 1 (by rotating, every loop is homotopic to one that fixes 1), then the path product h is a map that is f on the upper semicircle (1 to -1) and g on the lower (-1 to 1), and we have the following diagram

$$\begin{tikzcd} S^1 \arrow[r, "h"] \arrow[d] & S^1 \\ S^1 \wedge S^1 \arrow[ur, "f,g"'] & \end{tikzcd}$$

where the wedge is obtained by identifying $1, -1$ and giving both circles the anticlockwise orientation (orientation comes into picture only as a way to identify the two circles with the standard unit circle). Now, translate this to the homology groups. The vertical map sends 1 to $(1, 1)$, horizontal sends 1 to $\deg(h)$, and the diagonal sends $(1, 0)$ to $\deg(f)$, because of the choice of orientation and $(0, 1)$ to $\deg(g)$ for the same reason. Note that we are taking the generator of $H_1(S^1)$ to be the identity map, in the anticlockwise direction.

Therefore $\deg(h) = \deg(f) + \deg(g)$ and we are done.

References

[1] Hatcher, Algebraic Topology

[2] Spanier, Algebraic Topology