Periodicity in Complex K-theory

Shrivathsa Pandelu

November 7, 2023

Contents

1 Projectivization and the Canonical Bundle **2**

2 The Periodicity Theorem **3**

3 Fourier Variable **5**

4 Fourier Coefficients **7**
4.1 Neighbourhoods of sections contain metric neighbourhoods 9

5 Linearization **10**

6 Linear Clutching Functions **12**
6.1 Linear Algebraic Interlude 12
6.2 Decomposition I 13
6.3 Decomposition II 15

7 Finishing the proof **18**

8 Conclusion **21**

The goal of this article is to provide a more or less self contained proof of the periodicity theorem in complex K-theory. Our main references are Atiyah's K-theory and the notes by Wirthmüller on and vector bundles and K-theory.

Sections of a bundle $p\colon E \to X$ will be denoted by ΓE and the fibre over $x \in X$ will be denoted E_x. For statements about pullbacks, and operations of vector bundles (such as direct sum, tensor products et c.) refer [1]. Our base spaces are going to be compact, Hausdorff spaces. This, in particular, allows for partitions of unity and hence metrics on complex vector bundles.

Definition. *For a complex vector bundle $E \to X$, a metric on E is a pointwise poitive definite section $h\colon X \to \operatorname{Herm} E$ where $\operatorname{Herm} E$ is the (real) bundle whose fibres are Hermitian forms $E_x \times E_x \to \mathbb{C}$.*

Using partitions of unity, we can always provide E with a metric. A metric gives a norm and distance function on each E_x.

A sum of bundles indicates their direct sum.

1 Projectivization and the Canonical Bundle

For a space X, $\operatorname{Vect}(X)$ denotes the semiring of vector bundles on X and $K(X)$ the corresponding ring. For a bundle E on X, the zero section is given by the inclusion $X \hookrightarrow E$. Its complement is denoted by E_0. The complex numbers act on E_0 in the usual manner and the quotient space $P(E)$ is called the projectivization of E or the projective bundle associated to E.

$P(E)$ is a bundle over X because locally if $E|_U \cong U \times V$, then $P(E)|_U \cong U \times P(V)$ as the action of $\mathbb{C}^\times$ is fibre-preserving. Choosing a different trivialization gives an automorphism of the projective space because the transition before projectivization is some smoothly varying matrix function. For the same reason the quotient topology on $P(E)|_U$ is the same as that on $P(E|_U)$. Note also that the projection $E \to X$ factors through $P(E) \to X$.

Let V be any vector space and W a one dimensional space, then $V \cong V \otimes W$ by tensoring with any $w \in W$. This however is not a canonical isomorphism, although upon projectivization we get a canonical isomorphism $P(V) \cong P(V \otimes W)$. This result extends to vector bundles as follows.

Let E be any bundle, L a line bundle. If U is an open set where both trivialize, say $E|_U = U \times V, L|_U = U \times W$ then we have a local isomorphism as above which gives

$$P(E)|_U \cong U \times P(V) \cong U \times P(V \otimes W) \cong P(E \times L)|_U.$$

Let U' be another neighbourhood where E, L trivialize. On the intersection $U \cap U'$, the vectors of L are scaled by some nonzero complex functions when comparing the trivializations. This function disappears when we projectivize and the isomorphisms agree on the intersection. Therefore, these isomorphisms patch up giving us

$$P(E) \cong P(E \otimes L).$$

We have the following pullback square

$$\begin{tikzcd} p^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ P(E) \arrow[r, "p"'] & X \end{tikzcd}$$

Each point $a \in P(E)_x = P(E_x), x \in X$ defines a one dimensional subspace $H_x^* \subseteq E_x$. The union of these gives a subbundle of p^*E. It is indeed a subbundle because locally it works like a Grassmannian. This subbundle is the canonical line bundle H^*, dual of a bundle H.

2 The Periodicity Theorem

The rest of this article deals with the proof of

Theorem 1. *Let L be a line bundle over a compact space X. Then as a $K(X)$-algebra, $K(P(L \oplus 1))$ is generated by $[H]$ subject to the relation $([H] - [1])([L][H] - [1]) = 0$.*

Note here that $[L][H]$ is the product of $[L] \in K(X)$ with the element $[H]$ in the $K(X)$-algebra $K(P(L \oplus 1))$ (which is an algebra by pulling back bundles over X).

Remark. Line bundles are invertible in $K(X)$, but the same is not true in algebraic geometry.

The fibres of $P(L \oplus 1)$ are complex circles and we can embed L_x into this by $y \mapsto [y \oplus 1] \in P(L \oplus 1)_x$. This inclusion extends to a homeomorphism of the 1-point compactification of the fibres L_x (it is a continuous bijection from a compact space to a Hausdorff space, ergo a homeomorphism).

Each fibre has a point at infinity ∞_x and this gives a section, called the section at infinity, given by $x \mapsto \infty_x$. This is smooth because, locally, we choose a "constant" frame v for L and map x to $[v \oplus 0] \in P(L \oplus 1)_x$ (because L is a line bundle, a different trivialization gives a scaling factor which can be dropped when projectivizing). Similarly, there is a zero section.

The plan is as follows. We want the K-theory of $P(L \oplus 1)$. Intuitively, a line bundle contracts to X and has the same K-theory as X, except line bundles can twist in ways that prevent a contraction. Adding the $\mathbb{C}$ and projectivizing, in some sense, transfers this twist to the trivial bundle. The projectivization consists of (Riemann) sphere fibres which are the compactifications of contractible sets, i.e., complex planes (or 1 dimensional spaces). So we do something similar to the Van Kampen theorem, were we divide the projectivization into three parts, two contractible bits and their intersection which is a circle.

First choose a metric on L. Let $S \subseteq L$ be the unit circle bundle, P^0 the vectors with length ≤ 1 and P^∞ the part of $P(L \oplus 1)$ consisting of vectors of length ≥ 1 (perhaps a better description is "slope" ≥ 1) together with the section at infinity. Restricting $P(L \oplus 1) \to X$ gives projections

$$S \xrightarrow{\pi} X, P^0 \xrightarrow{\pi_0} X, P^\infty \xrightarrow{\pi_\infty} X.$$

Here π_0 is a homotopy equivalence because we can scale everything down to the zero section which is homeomorphic to X. π_∞ is also a homotopy equivalence because we can scale the other coordinate, i.e., the trivial coordinate coming from $\mathbb{C}$ to 0 to get a contraction to the section at infinity (which is again homeomorphic to X). This is done in patches and because we are only multiplying one of the coordinates by scalars, the different patches glue well.

So, the bundles over P^0, P^∞ are of the form $\pi_0^* E^0, \pi_\infty^* E^\infty$ respectively, for bundles E^0, E^∞ over X. Therefore, up to isomorphism, bundles over $P(L \oplus 1)$ are of the form $(\pi_0^* E^0, f, \pi_\infty^* E^\infty)$ where $f \in \mathrm{Iso}(\pi^* E^0, \pi^* E^\infty)$ is a clutching function. Moreover, for a given E^0, E^∞, the glued bundle E depends only on the homotopy class of f (Lemma 1.4.6 of [1]).

Suppose we are given a bundle E over $P(L \oplus 1)$, then we can obtain E^0, E^∞ by restricting E to the zero and infinity sections respectively. Because P^0, P^∞ deformation retract to these sections, there are isomorphisms

$$\varphi_0 \colon \pi_0^* E^0 \cong E|_{P^0}, \quad \varphi_\infty \colon \pi_\infty^* E^\infty \cong E|_{P^\infty}$$

from which we obtain f by a composition of these isomorphisms over S. Together with the clutching function we have an isomorphism $\varphi \colon (\pi_0^* E^0, f, \pi_\infty^* E^\infty) \xrightarrow{\cong} E$ that's the identity morphism at the zero and infinity sections.

Suppose we had another clutching function g and an isomorphism $\psi \colon (\pi_0^* E^0, g, \pi_\infty^* E^\infty) \xrightarrow{\cong} E$ which is also identity on the zero and infinity sections. Let ψ_0, ψ_∞ be the two parts of ψ as for φ.

Then $g = \psi_\infty \circ \psi_0^{-1}, f = \varphi_\infty \circ \varphi_0^{-1}$ as isomorphisms $\pi^* E^0 \to \pi^* E^\infty$. We have $\psi_0 \circ \varphi_0^{-1} \colon \pi^* E^0 \to \pi^* E^0$ and the following homotopy

$$\begin{aligned} H_t \colon \pi^* E^0 &\to \pi^* E^0 \\ (x, [v:1], w) &\mapsto (x, [v:1], (\psi_0 \circ \varphi_0^{-1})_{(x,[tv:1])}(w)) \end{aligned}$$

where $x \in X$, $[v:1] \in S_x$ so that for $0 \le t \le 1$, $[tv:1] \in P^0$ and $w \in E^0_x$ (the pulled back fibre over S_x is constant). This is a well defined homotopy of sections over S to the bundle $\mathrm{Hom}(\pi^* E^0, \pi^* E^\infty) = \pi^* \mathrm{Hom}(E^0, E^0)$ (homomorphisms are a continuous functor, so pullback behaves nicely; see Section 1.2 [1]). Note that the section $\psi_0 \circ \varphi_0^{-1}$ extends to the disc and this homotopy simply shrinks the circle through the disc to the zero section.

Therefore, $\psi_0 \circ \varphi_0^{-1}$ on the circle bundle is homotopic to the identity (because of the "end point" assumptions on φ, ψ). Doing the same for the other disc bundle, we see that $g^{-1} f$ is homotopic to the identity. This means that f is homotopic to g.

Therefore, if we force the isomorphism $E \to (\pi_0^* E^0, f, \pi_\infty^* E^\infty)$ to be identity on the end points, then the homotopy class of f is determined by the isomorphism class of E. Moreover, observe that each H_t is a clutching function, so two clutching functions f, g as above are not only homotopic, but they are homotopic through clutching functions.

Lemma 1. *Given a bundle E over $P(L \oplus 1)$, if we fix the restrictions E^0, E^∞ on the zero and infinity sections, then all the clutching functions are homotopic to each other, moreover the homotopy can be chosen to pass through clutching functions.*

Thus, we have a bijection $E \leftrightarrow (\pi_0^* E^0, f, \pi_\infty^* E^\infty)$ and we write (E^0, f, E^∞) for the latter. Our goal will be to control f or, more properly, its homotopy class. In analysis, we can approximate arbitrary continuous functions using their Fourier series (under some conditions), so we hope to do the same here.

3 Fourier Variable

On S we have two bundles and f is an isomorphism between them. Fibrewise, f is a function from the circle to a group of matrices. Under suitable assumptions, a function on S^1 is approximated by polynomials. For simplicity, suppose L is trivial, then $S = X \times S^1$ is trivial, and over the fibres we would have functions of the form $\sum a_k(x)z^k$ where n ranges over some finite subset of $\mathbb{Z}$, the a_k are matrix functions and would correspond to bundle morphisms between E^0, E^∞.

As a morphism, z is just scalar multiplication by the second coordinate of $(x, z) \in X \times S^1$, and this multiplication acts on the bundle E^0, say. There are two things to take care of. Firstly, S^1 here is topological, but has a multiplicative structure coming from the complex numbers. So, this multiplication works by "seeing" S^1 in $\mathbb{C}$, in other words, by getting a section $X \times S^1 \to X \times S^1 \times \mathbb{C}$ (there's no projection $X \times \mathbb{C} \to X \times S^1$) and the latter is the pullback of the trivial bundle over X to $X \times S^1$.

Secondly, multiplication on bundles needs to be clarified. There is a well defined product if we were to consider the trivial bundle, however, for a general L, the (fibre-wise) multiplication is more formal giving an element of $\pi^*L \otimes \pi^*E^0$ (because everything is happening over S).

Having clarified what z does, what should z^k be interpreted as? Although it is a section, the very operation of taking a kth power comes from being a section for $X \times S^1 \times \mathbb{C}^{\otimes k}$ and then composing with the k-fold multiplication. In the general case, z^k should be a section to a bundle which resembles $L^k = L^{\otimes k}$ and multiplication by z^k would result in an element of $\pi^*(L^k \otimes E^0)$.

The negative powers of z would then correspond to the tensor of the dual L^* of L. Keep in mind that even for vector bundles, taking dual and tensor powers commute and $L^k \otimes L^{k'} \cong L^{k+k'}$ for integers k, k' by patching up local isomorphisms.

Finally, what's the role of $a_k(x)$? Since z^k gives elements of $\pi^*(L^k \otimes E^0)$, a_k should be a section of $\mathrm{Hom}(L^k \otimes E^0, E^\infty)$. But we should be careful to interpret it as a section over S rather than over X, so properly it is a pullback of a section over X.

Once we have all this motivation down, because things are fibre-wise nice enough, we can use Fourier approximations to control what f should look like.

Bringing everything together, we have two bundles S, L over X. The pullback $\pi^*(L)$ is a subset of $S \times L$ and the diagonal $S \to S \times S \subseteq S \times L$ gives a section of $\pi^*(L)$ and denote this by z. Taking tensor products of the pullback and tensoring the section appropriately, we get sections z^k of $(\pi^*(L))^k$ for $k > 0$ and similarly z^{-k} of $(\pi^*(L^*))^k$.

For a section $a_k \in \Gamma\,\mathrm{Hom}(L^k \otimes E^0, E^\infty), \pi^*a_k$ is a section of $\mathrm{Hom}(\pi^*L^k \otimes \pi^*E^0, \pi^*E^\infty)$ over S and the product with z^k gives an element of

$$a_k z^k \stackrel{\text{def}}{=} \pi^*a_k \otimes z^k \in \Gamma(\mathrm{Hom}(\pi^*L^k \otimes \pi^*E^0, \pi^*E^\infty) \otimes \pi^*L^k) \cong \Gamma\,\mathrm{Hom}(\pi^*E^0, \pi^*E^\infty)$$

where the isomorphism is from the triviality of π^*L over S (and pullbacks commute with tensor products).

If a function $f = \sum_{\text{finite}} a_k z^k$ as above is a clutching function, then it is called a *Laurent clutching function*.

Proposition 1. *$H^* \cong (1, z, L)$.*

Proof. The pullback of the trivial bundle is $S \times \mathbb{C}$. Compose $z \times id_{\mathbb{C}}$ with product to get the clutching function

$$S \times \mathbb{C} \to \pi^*L \times \mathbb{C} \to \pi^*L.$$

This is a clutching function because a vector in π^*L can be scaled to have unit norm.

With that aside, recall that H^* is a subbundle of $p^*(L \oplus 1)$ where p is the projection $P(L \oplus 1) \to X$. We will figure out what the corresponding H^0, H^∞ and clutching functions are. For each $y \in P(L \oplus 1)_x$, the fibre H^*_y is the subspace of $(L \oplus 1)_x$ determined by y. Therefore,

$$H^*_{\infty,x} = L_x \oplus 0$$
$$H^*_{0,x} = 0 \oplus 1_x$$

where the subscript ∞, x denotes the ∞ over x. and similarly $0, x$.

Crucially, between line bundles if there is a morphism that is fibrewise nonzero, then it is an isomorphism, therefore if we were to follow the inclusion $H^* \hookrightarrow p^*(L \oplus 1)$ with projections to $p^*(1), p^*(L)$ (using pullback of projections from the direct sum), we get isomorphisms

$$f_0 \colon H^*|_{P^0} \to \pi_0^*(1); f_\infty \colon H^*|_{P^\infty} \to \pi_\infty^*(L)$$

(recall that π_0 is the restriction of p to the zero section and similarly π_∞). Note that we are projecting onto the nonzero coordinate.

So, H^* is built from clutching 1 and L using a function $f = f_\infty f_0^{-1} \colon \pi^*(1) \to \pi^*(L)$ (recall that π is the restriction of π_0, π_∞ to S). All that's left is to figure out what f is.

For $y \in S_x, H_y^*$ is spanned by $y \oplus 1 \in L_x \oplus 1_x$ and because the effect of f is to "go from one coordinate to the other" in $y \oplus 1$, the map $f \colon \pi^*(1) = S \times \mathbb{C} \to \pi^*(L)$ is precisely z.

More specifically, for $y \in S_x$ seeing $S_x \subset L_x$, if $\lambda \in \mathbb{C}_y$, then $f_0^{-1}(\lambda) = \lambda(y \oplus 1)$ and projecting it to the other coordinate gives $\lambda y \in L_x$; which in the pullback of L over y is precisely what $z(y, \lambda)$ gives. Therefore, $H^* \cong (1, z, L)$. $\square$

Because dual and powers happen fibrewise and clutching functions respond appropriately (it is, after all, a way to identify two bundles and hence must change accordingly), we see that

$$H^k \cong (1, z^{-k}, L^{-k}), k \in \mathbb{Z}.$$

4 Fourier Coefficients

Recall that when S^1 is parametrized by the interval $[0, 2\pi]$, the Fourier coefficients of a continuous (complex valued) function f are defined by $f_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}dt$ and the Fourier series is given by $\hat{f}(t) = \sum_{n\in\mathbb{Z}} f_k e^{ikt}$.

Change the variables so that we deal with a complex circle as is. Then the normalized measure on S^1 is $dz/(2\pi iz)$ and the Fourier coefficients are given by $f_k = \frac{1}{2\pi i}\int_{S^1} f(z)z^{-k-1}dz$. We are going to be computing the Fourier coefficients over the fibres of S and hence it is better to have a definition on S^1 rather than on $[0, 2\pi]$.

The setup so far: we have vector bundles E^0, E^∞ on X and fibre bundles $\pi\colon S \to X, \pi_0\colon P^0 \to X, \pi_\infty\colon P^\infty \to X$. Suppose $f \in \Gamma\,\mathrm{Hom}(\pi^*E^0, \pi^*E^\infty)$. This is not necessarily a clutching function.

On each fibre S_x of S over $x \in X$, this gives a function from the circle to a set of matrices. The role of z in the Fourier coefficient above should be played by the corresponding circle variable in S_x as seen in the complex plane over x, i.e., should be seen by the restricted section $z_x\colon S_x \to L_x$. The powers are to be similarly taken as the restricted section z_x^k.

What would $f_x(z)z_x^{-k-1}dz_x$, then be? Multiplying by z shifts E^0 to $L \otimes E^0$, so the product $f_x(z)z_x^{-k-1}dz_x$ would be a section of $\mathrm{Hom}(\pi^*L^k \otimes \pi^*E^0, \pi^*E^\infty)$ (with one more degree coming from dz_x; this is clear upon a change of variables, for example, or by interpreting dz_x as a section to L, i.e., a differential on S_x with entries in L_x.).

On the fibre, S_x, the pullback of E^0, E^∞, L are just product bundles and thus, $f_x z_x^{-k-1}dz_x$ is a map $S_x \to \mathrm{Hom}(L_x^k \otimes E_x^0, E_x^\infty)$. Integrating this (using the measure from the fibre L_x) over S_x gives the Fourier coefficients of $f, a_k(x)$ at x given by

$$a_k(x) = \frac{1}{2\pi i}\int_{S_x} f_x z_x^{-k-1}dz_x.$$

This gives $a_k \in \Gamma\,\mathrm{Hom}(L^k \otimes E^0, E^\infty)$. To see that it is well defined and continuous et c., we need local computations.

Starting with a neighbourhood where all the bundles in question trivialize, the definition of a_k is independent of the trivialization of L because of the factor dz/z and the scaling of z^{-k} term is cancelled by the corresponding scaling in the domain $L^k \otimes E^0$. The trivializations of E^0, E^∞ affect the matrix function f by change of basis matrices which freely move through the integral. Therefore, the local definition is independent of trivialization, hence well defined and clearly continuous. Thus a_k is indeed a section as mentioned above.

Multiplying a_k by z^k sends the domain back to π^*E^0. Define the (symmetric) partial sums

$$S_n = \sum_{-n}^{n} a_k z^k \in \Gamma\,\mathrm{Hom}(\pi^*E^0, \pi^*E^\infty).$$

The unfortunate reality that every analyst must come to terms with is that Fourier series don't always converge to the function, in fact, the Fourier series need not even be summable. However, the analyst need not despair for other modifications of the Fourier series do converge, usually.

Brief recap of Cesàro summation
The result is that if f is a continuous function on S^1, then the Cesàro sums converge to f uniformly. This is because the Fejér kernels

$$F_n(z) = \frac{1}{n+1} \sum_{k=0}^{n} \sum_{j=-k}^{k} z^j$$

form an approximation to identity, i.e., they satisfy

- $F_n \geq 0$
- $\frac{1}{2\pi i} \int_{S^1} F_n(z) \frac{dz}{z} = 1$
- $\forall\, \delta > 0, \lim_{n\to\infty} \int_{|z|>\delta} F_n(z) \frac{dz}{z} = 0$

The first two conditions together say that $||F_n||_1 = 1$. Intuitively, it's a sequence of functions that approach a Dirac delta function, which, when convolved with any function gives out the value of the function.
Convolving with an approximation to identity gives a sequence of functions which, by the defining conditions of an approximation to identity, converge to the function uniformly. For a proof that the Fejér kernels form an approximation to identity, see [2].

Set the Cesàro sums $f_n = \frac{1}{n} \sum_0^{n-1} S_k$, the average of the partial sums. We want to say that these converge to f uniformly. We have sections f_n, f of $\mathrm{Hom}(\pi^* E^0, \pi^* E^\infty)$. This is a bundle over S and f is a section.

Lemma 2. *Let $p\colon E \to X$ be a fibre bundle with fibre F. If X, F are quasicompact, then so is E.*

Proof. In this case, the projection p is a closed map. Suppose $C_i \subseteq E$ is a collection of closed sets with the finite intersection property, then $p(C_i)$ has the finite intersection property. Therefore, there is an $x \in \bigcap p(C_i)$. In the fibre F_x over $x, C_i \cap F_x$ is a collection of closed subsets of F_x with finite intersection property, hence nonempty intersection and that completes the proof. Observe that we didn't need E to be a fibre bundle, just that p is a perfect map (i.e., inverse of points are compact). It is easy to see that when X, F are Hausdorff, so is E. $\square$

Remark. The argument in the proof proves that a perfect map is proper.

Thus, S is a compact Hausdorff space, hence allows partitions of unity. This means that we can choose a metric on $\mathrm{Hom}(\pi^* E^0, \pi^* E^\infty)$. Although this feels arbitrary, it is reasonable to talk about convergence using a metric (see below).

Having chosen a metric on the bundle, fix an $\epsilon > 0$. Locally we have sections of the form

$$f, f_n \colon U \times S^1 \to U \times S^1 \times F$$

where F is some space of matrices. Although we have an extra parameter coming from U, the computations in showing that the Cesàro sums converge uniformly follow through just the same; the extra parameter tags along and doesn't affect anything. This is because all the computations happen at the same time on all the fibres over U (see [2]). Therefore, we can find some sufficiently large n such that all f_n are within f (fibrewise) over $U \times S^1$.

Covering X with such trivializing neighbourhoods and using compactness, we obtain some sufficiently large N such that f_n is within ϵ of f throughout S.

Lemma 3. *For sufficiently large n, f_n is a clutching function and $(E^0, f, E^\infty) \cong (E^0, f_n, E^\infty)$.*

Proof. The set $\mathrm{Iso}(\pi^* E^0, \pi^* E^\infty)$ of isomorphisms is an open neighbourhood of f in the bundle $\mathrm{Hom}(\pi^* E^0, \pi^* E^\infty)$. Therefore, there is some $\epsilon > 0$ such that anything that is (fibrewise) within ϵ of f is an isomorphism. So, for sufficiently large n, f_n is a clutching function.

Moreover, the straight line homotopy $tf_n + (1-t)f$ is a homotopy between f, f_n through isomorphisms, hence the resulting glued bundles are isomorphic. $\square$

Therefore, we can replace any arbitrary clutching function by a Laurent clutching function. Next we look at how to simplify Laurent clutching functions.

4.1 Neighbourhoods of sections contain metric neighbourhoods

Suppose $p\colon E \to X$ is a vector bundle over a compact Hausdorff space X and $f\colon X \to E$ is a section. Suppose N is a neighbourhood of $f(X)$. Choose any metric h on E. We wish to show that there is an $\epsilon > 0$ such that if $v \in E_x$ and $h_x(f(x), v_x) < \epsilon$ then $v \in N$.

First we reduce to the case where f is simply the zero section. We have the homeomorphism

$$\begin{aligned} E &\to E \\ v &\mapsto v - f(x) \end{aligned}$$

for $v \in E_x$. Its continuity is verified locally. The image of N is some open set N' containing the zero section. If we find an $\epsilon > 0$ as above for N' and the zero section, then the same ϵ works for N and f. Thus, we assume that N is a neighbourhood of the zero section.

Secondly, we bound N. Consider the open unit disc neighbourhood of X defined as

$$B_1 = \{v \in E : h_{p(v)}(v, 0_{p(v)}) < 1\}.$$

This is indeed open because on a trivializing neighbourhood $U \times V$, the function $h_x(v, 0_x)$ is a continuous function by the continuity of the metric and evaluation $\operatorname{Herm} V \times V \times V \to \mathbb{C}$ (although the distance function has codomain $\mathbb{R}$). Define

$$g(x) = h_x(v, 0_x)\colon E \to \mathbb{R}_{\geq 0}.$$

Then $N \cap B_1$ is an open set containing the zero section. The closure $\overline{B_1}$ is a compact subset of E because each of the fibre is compact (this uses two facts: all norms are equivalent for finite dimensional vector spaces and for finite dimensional normed linear spaces, the closed unit ball is compact). The argument is exactly the same as in Lemma 2.

Therefore, the closure and boundaries of $N \cap B_1$ are compact. The restriction of g to $\partial(N \cap B_1)$ is continuous hence attains a minimum $\epsilon \geq 0$. For $v \in E_x \cap \partial(N \cap B_1)$ if $g(v) = 0$, then by positive definiteness of $h, v = 0$ but this is a contradiction as $N \cap B_1$ is a neighbourhood of the zero section. Therefore, the minimum attained is a positive number $\epsilon > 0$.

Now we claim that this $\epsilon > 0$ is as stated above. Suppose $v \in E_x$ is within ϵ of 0_x. By definition of ϵ, either v is in the interior of $N \cap B_1$, in which case there's nothing to prove, or v is in the exterior.

In the fibre E_x, look at the line $tv, t \in [0, 1]$ joining $0_x, v$. This is connected and once again, by definition of ϵ, it doesn't intersect the boundary $\partial(N \cap B_1)$. Because the interior and exterior are disjoint sets, this line must lie entirely in the interior or entirely in the exterior. Since 0_x is in the interior, we conclude that so is v and therefore, $v \in N \cap B_1$.

Thus, $B_\epsilon \subseteq N \cap B_1 \subseteq N$ and we have proved that these ϵ neighbourhoods of the zero section form a local basis for the zero section, and by extension, form a local basis for the image of any section.

So, while talking about the convergence of the Cesàro sums, it suffices to consider ϵ neighbourhoods as above. Note that the exact choice of the metric doesn't matter these ϵ neighbourhoods form a local basis regardless of the metric and provide convergence in the usual sense using neighbourhoods.

5 Linearization

Having reduced to the case of Laurent polynomials, we consider a polynomial clutching function $p = \sum_{k=0}^{n} a_k z^k$, where $a_k \in \mathrm{Hom}(L^k \otimes E^0, E^\infty)$ (a general Laurent polynomial is $z^{-n}p$ for some n). Once again, we borrow a trick from analysis, this time a linearization trick to solve higher order differential equations.

Recall that in solving an equation of the form $y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1y'+y=0$, introducing the derivatives as separate functions to be solved for reduces the problem from an nth order differential equation to n linear differential equations. We do the same thing here by introducing more bundles and having linear clutching functions between them.

Consider the homomorphism

$$\mathcal{L}^n(p) \colon \pi^*\left(\sum_{k=0}^{n} L^k \otimes E^0\right) \to \pi^*\left(E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0\right)$$

given by the matrix

$$\mathcal{L}^n(p) = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ -z & 1 & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{bmatrix}$$

The first equation is supposed to capture p as a linear polynomial, while the rest introduce relations (for example, the second column says that the second variable should be z times the first). Although $a_k, z, 1$ are bundle morphisms, the usual matrix manipulations works fine because all we are doing is defining the morphism on different components of the domain and codomain of $\mathcal{L}^n(p)$.

Note that $\mathcal{L}^n(p)$ is linear in the sense that it can be written as $az+b$ for a, b suitable bundle morphisms between the two big bundles. Moreover, $\mathcal{L}^n(p_1+p_2) = \mathcal{L}^n(p_1)+\mathcal{L}^n(p_2)$ and it is compatible with tensor products in the sense that $\mathcal{L}^n(id_{\pi^*D} \otimes p) = id_{\pi^*D} \otimes \mathcal{L}^n(p)$ for a bundle $D \to X$.

To simplify p, write $p_0 = p$ and p_{r+1} to be a polynomial function such that $zp_{r+1} = p_r(z) - p_r(0)$ (this is formal manipulation; it doesn't make sense to set $z = 0$ as $z, 0$ are two different sections on S). Then, the homomorphism $\mathcal{L}^n(p)$ can be rewritten as

$$\mathcal{L}^n(p) = \begin{bmatrix} 1 & p_1 & p_2 & \dots & p_n \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} p & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -z & 1 & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{bmatrix}$$

by algebraic manipulation. Therefore,

$$\mathcal{L}^n(p) = (1+N_1)(p \oplus 1)(1+N_2)$$

where N_1, N_2 are nilpotent bundle endomorphisms (N_2 looks like shifting of coordinates and N_1 sends everything to E^∞ and E^∞ to 0) and $p \oplus 1$ indicates a direct sum of bundle morphisms.

$$\begin{array}{ccc} \pi^*\left(\sum_{k=0}^{n} L^k \otimes E^0\right) & \xrightarrow{\mathcal{L}^n(p)} & \pi^*\left(E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0\right) \\ {\scriptstyle (1+N_2)}\big\downarrow & & \big\uparrow{\scriptstyle (1+N_1)} \\ \pi^*\left(\sum_{k=0}^{n} L^k \otimes E^0\right) & \xrightarrow[p\oplus 1]{} & \pi^*\left(E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0\right) \end{array}$$

When N is nilpotent, the bundle endomorphisms $1+tN$ are invertible for $0 \le t \le 1$ and provide a path between 1 and $1+N$; this also shows that $\mathcal{L}^n(p)$ is a clutching function, because $p \oplus 1$ is.

Proposition 2. *$\mathcal{L}^n(p)$ and $p \oplus 1$ define isomorphic bundles on $P(L \oplus 1)$.*

Proof. The function $(1+tN_1)(p\oplus 1)(1+tN_2)$ is a path through bundle isomorphisms on S between $\mathcal{L}^n(p)$ and $p\oplus 1$, hence the two clutching functions are homotopic and give rise to isomorphic bundles on $P(L \oplus 1)$. □

In terms of earlier notation,

$$(E^0, p, E^\infty) \oplus \left(\sum_{k=1}^{n} L^k \otimes E^0, 1, \sum_{k=1}^{n} L^k \otimes E^0 \right) \cong \left(\sum_{k=0}^{n} L^k \oplus E^0, \mathcal{L}^n(p), E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0 \right). \quad (1)$$

Following [1], denote by $\mathcal{L}^n(E^0, p, E^\infty)$ the bundle on the right hand side above. As long as we stay with isomorphisms, we can homotope individual rows of $\mathcal{L}^n(p)$ to get potentially simpler matrices.

Lemma 4. *Let p be a polynomial clutching function of degree $\leq n$ for (E^0, E^∞). Then*

1. $\mathcal{L}^{n+1}(E^0, p, E^\infty) \cong \mathcal{L}^n(E^0, p, E^\infty) \oplus (L^{n+1} \otimes E^0, 1, L^{n+1} \otimes E^0)$
2. $\mathcal{L}^{n+1}(L^{-1} \otimes E^0, zp, E^\infty) \cong \mathcal{L}^n(E^0, p, E^\infty) \oplus (L^{-1} \otimes E^0, z, E^0)$

Note that in the definition of $\mathcal{L}^n(p)$ and the arguments above we did not require $a_n \neq 0$, only that p be a clutching function.

Proof sketch. The proof is fairly straightforward. In both cases, homotope a one of the rows through isomorphisms to get to a direct sum of matrices that act as clutching functions on the right. One thing to note is that in the second case, there is a homotopy

$$\mathcal{L}^{n+1}(zp) \simeq \mathcal{L}^n(p) \oplus (-z)$$

and $(L^{-1} \otimes E^0, -z, E^0) \cong (L^{-1} \otimes E^0, z, E^0)$ because $-z$ is the composition of z with -1, and -1 extends to all of E^0, hence is invertible (the E^0 on the right is a negative copy of E^0 on the left). The degree condition on p is important for the homotopies to work. □

Proposition 3. *For any polynomial clutching function p for (E^0, E^∞),*

$$([E^0, p, E^\infty] - [E^0, 1, E^0])([L][H] - [1]) = 0$$

in $K(P(L \oplus 1))$ where $[E^0, p, E^\infty]$ represents (E^0, p, E^∞) in $K(P(L \oplus 1))$.

Proof. From the second part of the last lemma and the linearization result before that,

$$(L^{-1} \otimes E^0, zp, E^\infty) \oplus \left(\sum_{k=0}^{n} L^k \otimes E^0, 1, \sum_{k=0}^{n} L^k \otimes E^0 \right) \cong$$
$$(E^0, p, E^\infty) \oplus \left(\sum_{k=1}^{n} L^k \otimes E^0, 1, \sum_{k=1}^{n} L^k \otimes E^0 \right) \oplus (L^{-1} \otimes E^0, z, E^0)$$

This essentially amounts to homotoping the same matrix (that of zp) to two different matrices. Passing everything to the K group, and ignoring the repeated terms,

$$[L^{-1} \otimes E^0, zp, E^\infty] \oplus [E^0, 1, E^0] = [E^0, p, E^\infty] \oplus [L^{-1} \otimes E^0, z, E^0].$$

Now, $zp = z \otimes p$, a clutching function between tensor product of bundles, similarly the z on the right hand side. Recall that $[1, z, L] = [H^{-1}]$, hence $[L^{-1}, z, 1] = [L^{-1}][H^{-1}]$ (using $L^{-1} \otimes L \cong 1$ and tensoring the clutching function with identity). Therefore,

$$[L^{-1}][H^{-1}][E^0, p, E^\infty] \oplus [E^0, 1, E^0] = [E^0, p, E^\infty] + [L^{-1}][H^{-1}][E^0, 1, E^0].$$

Rearranging the terms gives the desired relation. Keep in mind that $K(P(L\oplus 1))$ is commutative. □

In particular, for $E^0 = 1, p = z, E^\infty = L$, $([H] - [1])([L][H] - [1]) = 0$.

6 Linear Clutching Functions

We have a way of reducing a polynomial clutching function into a linear one, albeit on a different set of bundles. Here we analyse linear clutching functions of the form $az + b\colon \pi^* E^0 \to \pi^* E^\infty$.

How should one try to simplify this further? The simpler cases are when $a = 0$ or $b = 0$; although there is still a dependence on what E^0, E^∞ can be. But we could try to homotope $az + b$ to az or b by looking at $a(tz) + b$ or $az + tb = t(a(z/t) + b)$ for $0 \leq t \leq 1$. These need not be isomorphisms, however.

This boils down to extending $az+b$ to all of L (and, in a way, it extends as a at the infinity section - although the domain would be incorrect). To extend to L, we need to pullback E^0, E^∞ to L via the projection $L \to X$ and the morphisms a, b are also pulled back. The section $z\colon S \to \pi^* L$ becomes a section $L \to L \times_X L$ (the fibred product is L pulled back over itself). Recall that this proceeds through the diagonal map $L \to L \times L$. Label this extended function by p as well.

Convention. Following [1], when we say a linear function $p = az + b$ is an isomorphism outside S we include that it is an isomorphism at the infinity section as well, i.e., a is an isomorphism.

Fix an $x \in X$ and move to the fibre over x, so that the only free variable is z and we have essentially reduced the problem to something over $\mathbb{C}$. Let us look at $atz+b$, which connects $az+b$ to b. Suppose b is an isomorphism for now. Let us also identify E^0, E^∞ as a single vector space E^0 - we need to be careful about how we identify them for there's at least a circle's worth of isomorphisms!

If there is a nonzero v such that $atzv + bv = 0$, we get $-1/tz$ is an eigenvalue of $b^{-1}a$ (t can't be zero as b is assumed nonsingular). Conversely, any nonzero eigenvalue λ of $b^{-1}a$ gives a vector in the nullspace of $atz + b$ (this amounts to finding a z, t with z in the circle, such that $tz\lambda = -1$).

Similarly, we get (by the invertibility of $az + b$, the eigenvalues can't be on the circle)

- If a is invertible, $a^{-1}b$ has all eigenvalues outside the unit circle, then $atz + b$ is invertible.
- If a is invertible, $a^{-1}b$ has all eigenvalues inside the unit circle, then $az + tb$ is invertible.
- If b is invertible, $b^{-1}a$ has all eigenvalues inside the circle, then $atz + b$ is invertible.
- If b is invertible, $b^{-1}a$ has all eigenvalues outside the circle, then $az + tb$ is invertible.

On the fibre over $x \in X$ these a, b don't vary and we are looking endomorphisms of a vector space and asking about where its eigenvalues are. Obviously, the eigenvalues don't have to be divided as nicely.

Observe that in $atzv + bv = 0$, replcaing v with $a^{-1}v$, we can shift the burden of nonsingularity on to a. So, we're interested in the eigenvalues of the matrices $a^{-1}b, ba^{-1}, b^{-1}a$, and ab^{-1}.

6.1 Linear Algebraic Interlude

Suppose $T\colon E \to E$ is an endomorphism of a finite dimensional complex vector space (with some fixed basis). We'd like to know which eigenvalues of T are inside some circle S and which are outside.

Recall from complex analysis that if $\alpha \in \mathbb{C} \setminus S$, to know if α is inside or outside S, all we need to do is look at the integral $\frac{1}{2\pi i}\int_S \frac{1}{z-\alpha}dz$. This is because, if the circle contains α, then there's no branch of $\log$, but if it is outside, there is a branch of $\log$ defined on a domain containing the circle.

More generally, as long as we can get a factor of $1/(z-\alpha)$ in the denominator of a function, the integration as above serves as an indicator function (this isn't quite true, but it will do; refer the residue theorem for a precise statement).

To partition the eigenvalues into those lying inside S and those lying outside, we could simply ask what the integral of $1/z - \alpha$ is over S as α ranges over the eigenvalues, provided none of them lie on S. But the residue theorem allows us to do this, more or less, at once, provided we get a rational function that has the factors $z - \alpha$ in the denominator.

The characteristic polynomial, i.e., $\det(z - T)$ has the eigenvalues of T as its roots, inverting it gives us what we want. But rather than invert a polynomial, we invert the matrix $z - T$.

Consider the operator

$$Q = \frac{1}{2\pi i}\int_S (z-T)^{-1}dz \colon E \to E$$

where $(z-T)^{-1}$ makes sense by choice of S and the integral happens entry-by-entry. Each entry of the matrix of Q is of the form

$$\pm\frac{1}{2\pi i}\int_S \frac{z^{n-1}+\dots}{\det(z-T)}dz$$

where the numerator is a degree $n-1$ polynomial. By the residue theorem, this is the sum of residues of the integrand at the eigenvalues inside S. Computing this directly is insanity, so we resort to simplifying T first.

Assume a basis of E is so chosen that T is in it's Jordan normal form,

$$T = \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_r \end{bmatrix}$$

where each T_i is of the form

$$T_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \end{bmatrix}$$

It suffices to compute Q on the diagonal blocks. Say $T = T_i, \lambda = \lambda_i$, then

$$(z-T)^{-1} = \begin{bmatrix} \frac{1}{(z-\lambda)} & \frac{1}{(z-\lambda)^2} & \dots & \dots \\ & \frac{1}{(z-\lambda)} & \frac{1}{(z-\lambda)^2} & \dots \\ & & \ddots & \ddots \\ & & & \frac{1}{(z-\lambda)} \end{bmatrix}$$

whose residues give I if λ is inside S and 0 otherwise.

It is now clear that Q is a projection operator, i.e., $Q^2 = Q$. If the Jordan normal form above corresponds to a decomposition of E as $E = E_1 \oplus \dots \oplus E_r$, then Q projects onto those E_i for which λ_i is inside the circle. Since T commutes with Q, it respects the decomposition offered by Q.

Set $E = E_+ \oplus E_-$ where $E_+ = QE, E_- = (I-Q)E$, then T decomposes as $T_+ \oplus T_-$. By construction of Q, the eigenvalues of T_+ are inside S and those of T_- are outside.

6.2 Decomposition I

The reader should feel free to skip to the next subsection to continue with the proof of the periodicity theorem. Let's focus on the fibre over $x \in X$. Here we have

- two vector spaces E^0_x, E^∞_x
- homomorphisms given by $p(z) = az + b, z \in \mathbb{C}$ which are isomorphisms for $z \in S^1$, hence on a neighbourhood of S^1.

Once we are in the fibre over x, a is simply a matrix because $L_x \otimes E^0_x \cong E^0_x$ via scalar multiplication. We also know that if we had a single endomorphism, $T \colon E \to E$, then we can decompose E, T as above so that T_+, T_- have eigenvalues in different regions.

From earlier discussion, if we could decompose E^0_x, E^∞_x so that $a^{-1}b, ba^{-1}$ (and/or $b^{-1}a, ab^{-1}$) decompose and if, simultaneously, all $az+b$ respect this decomposition, then we can use the homotopies $atz + b, az + tb$.

To decompose $a^{-1}b$, for example, we would have to look at $(z - a^{-1}b)^{-1} = (az + b)^{-1}a$. The left hand side is not defined, but the right certainly is for $z \in S^1$. Similarly, tackling the other three possible matrices, we're interested in the integrals (over the circle) of

$$(az + b)^{-1}a, a(az + b)^{-1}, (bz + a)^{-1}b, b(bz + a)^{-1}.$$

The first two of these are related to the linear function p and can be written as the operators

$$Q_x^0 = \frac{1}{2\pi i}\int_{S_x} p_x^{-1} dp_x \text{ and } Q_x^\infty = \frac{1}{2\pi i}\int_{S_x} dp_x p_x^{-1}$$

where the superscript indicates which of E_x^0, E_x^∞ do these act on. Here dp_x is simply the derivative of $az+b$ as a function from $\mathbb{C}$ to a space of matrices. The idea is that these would decompose E_x^0, E_x^∞ (i.e., they are projections) where p can be homotoped to something simpler.

For the other two, while $bz + a$ can make sense as a morphism $E_x^0 \to E_x^\infty$ after various non canonical identifications, to extend it as a global bundle morphism we must take $z \in L^{-1}$, the dual bundle, for then $bz + a$ can become a bundle morphism $L \otimes E^0 \to E^\infty$.

Abusing notation, we can have a diagram as follows,

$$\begin{tikzcd}
E_x^0 \arrow[r, "z\oplus 1"] \arrow[d, "z"'] & (L_x \otimes E_x^0) \oplus E_x^0 \arrow[r, "a\oplus b"] \arrow[d, "\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}"] & E_x^\infty \arrow[d, equal] \\
L_x \otimes E_x^0 \arrow[r, "\tilde{z}\oplus 1"'] & E_x^0 \oplus (L_x \otimes E_x^0) \arrow[r, "b\oplus a"'] & E_x^\infty
\end{tikzcd}$$

where $z \in L_x, \tilde{z} = 1/z \in L_x^{-1}$, and $a \oplus b$ is not really a direct sum of matrices, but composed with the addition in E_x^∞. The bottom row gives $b\tilde{z} + a$. By abusing notation again, we can write

$$b\tilde{z} + a = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{z} \\ 1 \end{pmatrix}.$$

Thus, $b\tilde{z} + a$ is obtained from $az + b$, up to a scaling factor, by a change of "variables" $z = 1/\tilde{z}$ and a change of domain. We put variables in quotes because z is not the variable that $az + b$ acts on. This change of variables is given by the action of a matrix on $z \oplus 1$ to give $1 \oplus z$ (in other words, it's not just a change of z, but of $z \oplus 1$ treated as one entity). Seeing $z \oplus 1$ as the line $[z : 1]$ in $\mathbb{C}^2$ (by identifying L_x with $\mathbb{C}$), this is the Möbius transformation determined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Keep in mind that this is more of a notational manipulation and doesn't extend to the whole of X. More importantly, this change of variables by inversion maps the circle to itself, but interchanges the components of the complement. As far as integration is concerned, we don't have to worry about some convoluted loop.

If we could change the variables in such a way that the coefficient of z became invertible, then we can use the result above to conclude that our operators are indeed projections and decompose $p(z)$ into parts whose eigenvalues are in different components of $\mathbb{C} \setminus S^1$.

Manipulation of z as above essentially does a linear transformation on $\begin{pmatrix} a & b \end{pmatrix}$. So pick an $\alpha \in \mathbb{C}$ (arbitrary for now) so that $a\alpha + b$ is an isomorphism. We can identify E_x^0, E_x^∞ using this, i.e., fix a basis of E_x^0, and let its image be the basis of E_x^∞. Then with respect to this identification, $a\alpha + b$ is the identity matrix.

In place of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we could choose a matrix of the form $\begin{pmatrix} \alpha & u \\ 1 & v \end{pmatrix}$ so that we get the invertible $a\alpha + b$ as the coefficient of z. The matrix acts via the Möbius transformation, and if one were to choose u, v so that the unit circle is mapped to itself, the simplest solution is

$$z \mapsto \frac{\alpha z + 1}{z + \alpha}$$

obtained by comparing the norms $|\alpha z+u|^2=|z+v|^2$ for $z\in S^1$.

The corresponding Möbius transformation works for α not in the circle, but this is fine because $az+b$ is an isomorphism on a neighbourhood of S^1.

Note to self
Although I have more or less motivated to myself Möbius transformations entering the conversation, it still feels a little inorganic; especially when I'm aware of the argument having read it from [1]. My train of thought is as follows

1. We want to know about the eigenvalues of $a^{-1}b$ et c. and that forces us to look at the integrals of $az+b, a+bz$.
2. But $a+bz$ doesn't make sense unless we change the domain to $L\otimes E^0$ and make z live in the dual space.
3. This change happens via the action of a matrix on $z\oplus 1$, which is a line in $\mathbb{C}^2$ provided we identify L_x with $\mathbb{C}$ on this particular fibre.
4. To use the result from linear algebra, we need the coefficient of z to be 1 so that we are dealing with actual characteristic polynomials.
5. So the change of variable should transform the line $z\oplus 1$ to another line, and such a transformation is aided by Möbius transformations, such that the matrix naturally gives $a\alpha+b$ for a suitable α for which it is an isomorphism.
6. For our own convenience, the domain better stay unchanged, and the simplest matrix that works is as above.

Note also that the Möbius transformation works to send α to infinity, which, in some sense, gives us $a^{-1}(az+b)$ from $(a\alpha+b)^{-1}(az+b)$.

6.3 Decomposition II

Proposition 4. *Let p be a linear clutching function for (E^0,E^∞), and define endomorphisms Q^0,Q^∞ of E^0,E^∞ by putting*

$$Q^0_x=\frac{1}{2\pi i}\int_{S_x}p_x^{-1}dp_x \quad Q^\infty_x=\frac{1}{2\pi i}\int_{S_x}dp_xp_x^{-1}.$$

Then Q^0,Q^∞ are projection operators, and

$$pQ^0=Q^\infty p.$$

Write $E^i_+=Q^iE^i, E^i_-=(1-Q^i)E^i, i=0,\infty$, so that $E^i=E^i_+\oplus E^i_-$. Then p is compatible with these decompositions, so that $p=p_+\oplus p_-$. Moreover, p_+ is an isomorphism outside S, p_- is an isomorphism inside S.

Although Q^0,Q^∞ are defined pointwise, in the local picture we would be conjugating p by matrix functions (change of basis matrices) that depend on $x\in X$ only, and hence the projections patch up. If we were dealing with manifolds, then d is the exterior derivative and since we're integrating over the fibres, only the z-derivative bit survives.

Proof. It suffices to prove the results fibre-wise. So, fix $x\in\mathbb{C}$, to simplify we denote p_x,a_x,b_x by p,a, and b respectively. Pick a real number $\alpha>1$ near S_x for which $p(\alpha)$ is an isomorphism. Consider the Möbius transformation $w=\frac{1-\alpha z}{z-\alpha}$ so that $z=\frac{1+\alpha w}{\alpha+w}$, then

$$p(w)=a\left(\frac{1+\alpha w}{\alpha+w}\right)+b=\frac{(a\alpha+b)w+(a+b\alpha)}{w+\alpha}=\frac{T_1w+T_2}{w+\alpha}$$

where $T_1 = a\alpha + b, T_2 = a + b\alpha$, and

$$dp = \frac{T_1\alpha - T_2}{(w+\alpha)^2} dw.$$

The integrand becomes

$$\begin{aligned} p^{-1}dp &= (w+\alpha)(w + T_1^{-1}T_2)^{-1}T_1^{-1}T_1 \frac{\alpha - T_1^{-1}T_2}{(w+\alpha)^2} dw \\ &= \frac{(w-T)^{-1}(\alpha+T)}{(w+\alpha)^2} \end{aligned}$$

where $T = T_1^{-1}T_2$. Writing $\alpha + T = (\alpha + w) - (w - T)$ gives

$$Q_x^0 = \frac{1}{2\pi i}\int_{|w|=1} ((w-T)^{-1} - (\alpha+w)^{-1})dw = \frac{1}{2\pi i}\int_{|w|=1} (w-T)^{-1}dw$$

where the last equality follows because α was chosen outside the circle. It is easy to verify that this T doesn't have any eigenvalues on the unit circle, so the integral makes sense and decomposes E_x^0 into two parts, and T respects this decomposition. Moreover, since T commutes with Q^0, so does $T_1^{-1}p(w)$.

In a similar way, Q_x^∞ is a projection operator on E_x^∞ and decomposes $T' = T_2T_1^{-1}$ and commutes with $p(w)T_1^{-1}$. Moreover,

$$T_1Q^0 = Q^\infty T_1$$

simply because the same relation holds for the integrand and T_1 can pass freely through the integrals. From this relation, we obtain

$$T_1^{-1}pQ^0 = Q^0T_1^{-1}p = T_1^{-1}Q^\infty p$$

and therefore, p decomposes as $p = p_+ \oplus p_-$ where

$$\begin{aligned} &p_+ : Q_x^0E_x^0 \to Q_x^\infty E_x^\infty \\ &p_- : (1 - Q_x^0)E_x^0 \to (1 - Q_x^\infty)E_x^\infty \end{aligned}$$

All of this is true for $p(w), w \in \mathbb{C}$. Continuing with the same notational convention, all eigenvalues of T_+, T'_+ are inside the unit circle, and those of T_-, T'_- are outside. Suppose for some $|w| > 1, p_+(w)$ is singular, i.e., there is some nonzero $v \in Q_x^0E_x^0$ such that $p_+(w)v = 0$, then

$$T_1^{-1}\frac{w+T}{w+\alpha}v = 0 \implies Tv = T_+v = -wv$$

which is a contradiction as all eigenvalues of T_+ are inside the unit circle. Similarly, p_- is an isomorphism inside the unit circle. □

Corollary 1. *Let $p = az + b$ and write*

$$p_+ = a_+z + b_+, p_- = a_-z + b_-.$$

Then, if $p(t) = p_+(t) \oplus p_-(t)$, where

$$p_+(t) = a_+z + tb_+, p_-(t) = ta_-z + b_-, 0 \le t \le 1,$$

we obtain a homotopy of linear clutching functions connecting p with $a_+z \oplus b_-$. Thus,

$$(E^0, p, E^\infty) \cong (E_+^0, z, L \otimes E_+^0) \oplus (E_-^0, 1, E_-^0).$$

Proof. Firstly, note that p_+ is the restriction of p followed by a projection, therefore it is still a linear function and of the form $a_+ z + b_+$ (since tensor distributes over the direct sum, a decomposes too).

From the proposition, $p_+(t), p_-(t)$ are isomorphisms on S for $0 \leq t \leq 1$ and are clearly continuous. Thus,

$$(E^0, p, E^\infty) = (E^0, p(1), E^\infty) \cong (E^0_+, a_+ z, E^\infty_+) \oplus (E^0_-, b_-, E^\infty_-)$$

where we have used the fact that clutching with an isomorphism that's a direct sum is the direct sum of the bundles clutched separately.

Since $a_+ \colon L \otimes E^0_+ \to E^\infty_+, b_- \colon E^0_- \to E^\infty_-$ are isomorphisms, we can replace the right hand side by the left (and change the clutching functions appropriately) to get the stated isomorphisms. $\square$

7 Finishing the proof

The rest of this document is pretty much the same as [1]. Having analysed a linear clutching function, let p be a clutching function of degree $\leq n$. Then $\mathcal{L}^n(p)$ is a linear clutching function for (V^0, V^∞) where

$$V^0 = \sum_{k=0}^{\infty} L^k \otimes E^0, \quad V^\infty = E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0.$$

We can decompose $V^0 = V^0_+ \oplus V^0_-$ as above. Write

$$V^0_+ = V_n(E^0, p, E^\infty).$$

This is a bundle on X. Now, suppose p_t is a homotopy through polynomial clutching functions of degree $\leq n$ on S. Extend (via pullback under projection) L, E^0, E^∞ to $X \times I$ (this is still compact Hausdorff!) so as to construct the bundle $V_n(E^0, p_t, E^\infty)$ over $X \times I$ (its construction requires p_t to be a clutching function for every t). The pullback of this V_n via the homotopic $0, 1$-inclusions $X \hookrightarrow X \times I$ are isomorphic (see Lemma 1.4.3 of [1])

$$V_n(E^0, p_0, E^\infty) \cong V_n(E^0, p_1, E^\infty).$$

In other words, if p_0, p_1 are homotopic through polynomial clutching functions, then the resulting bundles are isomorphic. Now recall the statement of Lemma 4 for p of degree $\leq n$:

1. $\mathcal{L}^{n+1}(E^0, p, E^\infty) \cong \mathcal{L}^n(E^0, p, E^\infty) \oplus (L^{n+1} \otimes E^0, 1, L^{n+1} \otimes E^0)$
2. $\mathcal{L}^{n+1}(L^{-1} \otimes E^0, zp, E^\infty) \cong \mathcal{L}^n(E^0, p, E^\infty) \oplus (L^{-1} \otimes E^0, z, E^0)$

where these bundles where obtained by clutching V^0, V^∞ using the linearized version of p.

Because the decomposition is via a projection which is given by an integral, it will respect the direct sums appearing on the right. Similarly, the isomorphism above sends the clutching function on the left to the one on the right, hence sends the projection operator on the left to the corresponding one on the right. Therefore, if we use the results from the previous section we get (for p of degree $\leq n$)

1. $V_{n+1}(E^0, p, E^\infty) \cong V_n(E^0, p, E^\infty)$,
2. $V_{n+1}(L^{-1} \otimes E^0, zp, E^\infty) \cong V_n(E^0, p, E^\infty) \oplus (L^{-1} \otimes E^0)$

where we have observed that if p is the identity function, then the projection Q^0 is zero. For the second isomorphism, we use that when $p = z, Q^0$ is identity.

Tensoring the second isomorphism with L we get

$$V_{n+1}(E^0, zp, L \otimes E^\infty) \cong (L \otimes V_n(E^0, p, E^\infty)) \oplus E^0$$

Recall Equation 1

$$(E^0, p, E^\infty) \oplus \left(\sum_{k=1}^{n} L^k \otimes E^0, 1, \sum_{k=1}^{n} L^k \otimes E^0 \right) \cong \left(\sum_{k=0}^{n} L^k \oplus E^0, \mathcal{L}^n(p), E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0 \right).$$

On the right side we have V^0, V^∞ and a linear clutching function, so we can apply Corollary 1 of the last section to get

$$(E^0, p, E^\infty) \oplus \left(\sum_{k=1}^{n} L^k \otimes E^0, 1, \sum_{k=1}^{n} L^k \otimes E^0 \right) \cong (V^0_+, z, L \otimes V^0_+) \oplus (V^0_-, 1, V^0_-).$$

Now we move to $K(P(L \oplus 1))$. Recall from Proposition 1 that $[H^{-1}] = [1, z, L]$. We get

$$[E^0, p, E^\infty] + \left\{ \sum_{k=1}^{n} [L^k \otimes E^0] \right\} [1] = [V_n(E^0, p, E^\infty)][H^{-1}] + \left\{ \sum_{k=0}^{n} [L^k \otimes E^0] - [V_n(E^0, p, E^\infty)] \right\} [1]$$

which gives

$$[E^0, p, E^\infty] = [V_n(E^0, p, E^\infty)]([H^{-1}] - [1]) + [E^0][1]$$

where the $[1]$ is the trivial bundle on $P(L\oplus 1)$ and $[E][1]$ for E a bundle over X is the pullback of E to $P(L \oplus 1)$. Importantly, the terms on the left are $K(X)$-multiples of two fixed terms in $K(P(L \oplus 1))$.

Proof of the Periodicity Theorem 1. Let t be an indeterminate over the ring $K(X)$ and consider $t \mapsto [H] \in K(P)$. From Proposition 3, $([H] - [1])([L][H] - [1]) = 0$, so we get a morphism of rings

$$\mu\colon K(X)[t]/((t-1)([L]t-1)) \to K(P).$$

We construct an explicit inverse. All the work we have done so far pretty much proves that μ is surjective.

First, suppose f is a clutching function for (E^0, E^∞). Let f_n be the associated Cesàro means and set $p_n = z^n f_n$. For sufficiently large n, p_n is a polynomial clutching function (of degree $\leq 2n$) for $(E^0, L^n \otimes E^\infty)$. For large n, we expect this to be $(E^0, f_n, E^\infty) \otimes H^{-n}$. Define

$$\nu_n(f) = [V_{2n}(E^0, p_n, L^n \otimes E^\infty)](t^{n-1} - t^n) + [E^0]t^n \in K(X)[t]/((t-1)([L]t-1)).$$

Since this is obtained by inverting H, for large n we expect this to be the inverse of $[E^0, f, E^\infty]$. So we prove that this is independent of n.

For sufficiently large n, f_n, f_{n+1} are close to f and are isomorphisms. Moreover, the convex linear combination of f_n, f_{n+1} is a homotopy through clutching functions. Therefore, the linear homotopy between p_{n+1}, zp_n is a polynomial homotopy through clutching functions of degree at most $2n+2$. Thus,

$$\begin{aligned} V_{2n+2}(E^0, p_{n+1}, L^{n+1} \otimes E^\infty) &\cong V_{2n+2}(E^0, zp_n, L^{n+1} \otimes E^\infty)(\text{ by the homotopy}) \\ &\cong V_{2n+1}(E^0, zp_n, L^{n+1} \otimes E^\infty)(zp_n \text{ is of degree } \leq 2n+1) \\ &\cong (L \otimes V_{2n}(E^0, p_n, L^n \otimes E^\infty)) \oplus E^0 \end{aligned}$$

Since $(t-1)([L]t-1) = 0$, we get $\nu_{n+1}(f) = \nu_n(f)$. So, $\nu_n(f)$ is eventually constant, call it $\nu(f)$. We claim that this is the inverse of μ.

Given a bundle E on $P(L \oplus 1)$ the homotopy class of its clutching function is uniquely determined. In the space of all sections over S to an appropriate homomorphism bundle (between the "ends" E^0, E^∞ of E), a homotopy of two clutching function f, g is a path.

Suppose, for the moment, that g is sufficiently close to f. Then for n sufficiently large, the linear interpolation between $z^n f_n, z^n g_n$ passes through polynomial clutching functions of degree $\leq 2n$. Therefore, the associated bundles $\nu_n(f), \nu_n(g)$ are going to be equal (recall that this requires the homotopy to pass through isomorphisms to make sense). Thus,

$$\nu(f) = \nu_n(f) = \nu_n(g) = \nu(g)$$

This means that ν is a locally constant function on the space of sections to the isomorphism bundle (which is an open set in the homomorphism bundle).

Going all the way back to Lemma 1, we see that $\nu(f)$ depends only on E because any two f with the same E are homotopic through clutching functions. Set $\nu(E) = \nu(f)$.

This is additive in E because the clutching functions are additive and so are the Cesàro means, linearization, and projections. Therefore we have an induced group homomorphism

$$\nu\colon K(P) \to K[X](t)/((t-1)([L]t-1)).$$

It's easy to see that this is a $K(X)$-module homomorphism because tensoring with bundles over X doesn't really affect the clutching functions and the tensor passes through the projection operators to give the same result as multiplication does on the right hand side. Finally,

$$
\begin{aligned}
\mu\nu(E) &= \mu[V_{2n}(E^0, p_n, L^n \otimes E^\infty)](t^{n-1} - t^n) + [E^0]t^n \\
&= [V_{2n}(E^0, p_n, L^n \otimes E^\infty)]([H]^{n-1} - [H]^n) + [E^0][H]^n \\
&= [E^0, p_n, L^n \otimes E^\infty][H]^n \\
&= [E^0, f_n, E^\infty] \\
&= [E^0, f, E^\infty] = E.
\end{aligned}
$$

Since $K(P)$ is generated by bundles $E, \mu\nu$ is identity. In the other direction, since we already have a $K(X)$-module homomorphism, it suffices to look at what happens to $t^n, n \geq 0$:

$$
\begin{aligned}
\nu\mu(t^n) &= \nu(H^n) \\
&= \nu[1, z^{-n}, L^{-n}] \\
&= [V_{2n}(1,1,1)](t^{n-1} - t^n) + [1]t^n \\
&= t^n
\end{aligned}
$$

where we observe that $V_{2n}(1,1,1) = 0$. $\square$

8 Conclusion

Immediate Corollaries

If X is a one point space, then all bundles are automatically trivial and $K(X) \cong \mathbb{Z}$. The space $P(L \oplus 1)$ is then just the 2-sphere S^2 and periodicity tells us

$$K(S^2) \cong \mathbb{Z}[t]/(t-1)^2.$$

Secondly, if L was the trivial bundle, then $P(L \oplus 1)$ is the trivial sphere bundle and we have an isomorphism

$$\mu\colon K(X) \otimes K(S^2) \to K(X \times S^2)$$
$$a \otimes b \to (\pi_1^* a)(\pi_2^* b)$$

where π_1, π_2 are the two projections. Compare this with the Künneth formula.

Equivariant Periodicity

Let G be a finite group and X a G-space. We have the set $\mathrm{Vect}_G(X)$ of G-vector bundles over X. This is also an abelian semigroup and we can form the group $K_G(X)$.

The projectivization of a G-bundle E will be a G-space (because G acts on the fibres of E in a linear manner). We can construct $P(L \oplus 1)$, for G-line bundles L as before and form the sub-G-bundle H over $P(L\oplus 1)$. The Fourier variable is a G-section, and all the integrals above are G-linear. It follows that the projections, approximations et c. respect the G-action and we get

Theorem 2. *If X is a G-space, L a G-line bundle over X, the map $t \mapsto [H]$ induces an isomorphism f $K_G(X)$-modules*

$$K_G(X)[t]/(t[L]-1)(t-1) \to K_G(P(L \oplus 1)).$$

Other Proofs

We have only touched on the complex case, there is also a real periodicity where the period is 8. These results were initially proved using Morse theory. The result is quite important and useful in many different contexts and appears in different forms. There have been multiple proofs of the periodicity theorem (both real and complex cases) and [5] is an interesting thread regarding these proofs.

References

[1] Atiyah, Michael Francis. K-theory. Netherlands, Benjamin, 1967.

[2] Wirthmüller, Klaus, Vector Bundles and K-Theory, Skeleton Lecture Notes 2011/2012 [pdf]

[3] Ahlfors, Lars Valerian. Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. India, McGraw-Hill, 1979.

[4] Stein, Elias M., and Shakarchi, Rami. Fourier analysis: an introduction. United Kingdom, Princeton University Press, 2003.

[5] Eric Peterson (https://mathoverflow.net/users/1094/eric-peterson), Proofs of Bott periodicity, URL (version: 2010-03-05): https://mathoverflow.net/q/8800