A brief introduction to topics in Morse Theory

Shrivathsa Pandelu

Indian Statistical Institute, Kolkata

18 November 2021

Abstract

In this lecture we present some topics from Morse Theory and show how Morse Theory helps in understanding the topology of the manifolds. Specifically we describe the stable and unstable manifolds associated to a Morse function and state the Morse theorems concluding with Reeb's theorem and the classic example of the vertical torus.

Morse functions

Given a smooth real valued function f on a manifold M, the critical points of f are those $p \in M$ where $df_p = 0$. If p is a critical point, then the Hessian of f at p is a well defined bilinear form on T_pM.The general Hessian is defined in the presence of a connection, however we are only interested in the Hessian of f at critical points. On $(U, x^1, \ldots, x^n)$ define:

$$H(f) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right)_{i,j}.$$

This is well defined at critical points and corresponds to the bilinear form $(v, w) \mapsto v(\tilde{w} f) = w(\tilde{v} f)$ where $\tilde{v}, \tilde{w}$ are local extensions of $v, w \in T_pM$. The critical point p is said to be non-degenerate if this Hessian is non-singular. f is said to be Morse if all its critical points are non-degenerate.

Morse Lemma

It is a theorem of linear algebra that any bilinear form can be diagonalized to the form

$$\begin{bmatrix} -I_r & & \\ & I_s & \\ & & 0 \end{bmatrix}$$

where r is called the index and $n - r - s$ is called the nullity. These numbers are uniquely defined. The proof is a version of the Gram-Schmidt algorithm. We can use a similar technique to prove

Lemma

(Morse lemma) Let p be a non-degenerate critical point of f. Then there is a chart $(U, y^1, \ldots, y^n)$ around p with $y^i(p) = 0 \forall i$ and

$$f = f(p) - (y^1)^2 - \cdots - (y^k)^2 + (y^{k+1})^2 + \cdots + (y^n)^2$$

on U where k is the index of f at p.

In general, given a quadratic form

$$Q = -(y^1)^2 - \cdots - (y^k)^2 + (y^{k+1})^2 + \cdots + (y^n)^2$$

we write it as

$$Q = -\|y_-\|^2 + \|y_+\|^2$$

where $y_- = (y^1, \ldots, y^k)$ and $y_+ = (y^{k+1}, \ldots, y^n)$.
As a corollary, observe that the non-degenerate critical points of f are isolated. If M is compact, then it follows that a Morse function f can have only finitely many critical points (this is not true for non-Morse functions: consider the height function on a torus lying on the plane).

Existence of Morse functions

Let M be a manifold embedded in $\mathbb{R}^n$. Using Sard's theorem it can be shown that for almost all $x_0 \in \mathbb{R}^n$, the function $x \mapsto \|x - x_0\|^2$ is a Morse function on M. Morse functions are also abundant. For example, in [2], we have the following theorem

Theorem

Let M be embedded in $\mathbb{R}^n$ and let $f : M \to \mathbb{R}$ be a smooth function. Let k be an integer. Then f and all its derivatives of order $\leq k$ can be uniformly approximated by Morse functions on every compact subset.

Pseudo-gradients

An important idea in Morse theory is that of flowing along the gradient of f. Intuitively, f describes a certain property of the space, say the height or energy and the flow describes how this quantity changes. This description is intimately tied to the shape of the space and Morse homology is a way to explore this connection. To obtain a gradient one can use a Riemannian metric as in [1]. However this is not always easy to use so instead of an exact gradient we use pseudo-gradients.

A pseudo-gradient for f is a vector field X on M such that

- $df(X) \leq 0$ everywhere with equality only at critical points..
- In a Morse chart around critical points, X agrees with the negative of the Euclidean gradient of f.

Existence of pseudo-gradient

As in the case of Riemannian metrics, we can construct pseudo-gradients using partitions of unity. Let M be compact, then the critical points are finite and isolated. We take a cover $\{U_i\}$ of M by charts such that each critical point is in a unique U_i.

On each chart we have a negative gradient of f and we add these up using a partition of unity. The important thing is that each critical point is in a single U_i, therefore this new vector field agrees with the negative gradient as required.

Local flow

In order to analyse the flow corresponding to any pseudo-gradient, we first look at the local picture. In a Morse chart, the pseudo-gradient X matches with the actual negative gradient (shrink U if necessary), so in coordinates we have

$$X = \sum_{i=1}^{k} 2x^i \partial_i + \sum_{i=k+1}^{n} -2x^i \partial_i$$

Then by the uniqueness of solutions to ODEs, the flow starting from $(c^1, \ldots, c^n) \in U$ is given by

$$\gamma : t \mapsto (c^1 e^{2t}, \ldots, c^k e^{2t}, c^{k+1} e^{-2t}, \ldots, c^n e^{-2t}) \tag{1}$$

as long as it lies in U.
By construction, f is strictly decreasing along flow lines (unless we are talking about a critical point).

The Standard ball

With Q as before, the standard balls around p is defined as

$$U(\epsilon,\eta) = \{x : |Q(x)| \leq \epsilon, \|x_-\|^2 \|x_+\|^2 \leq \eta(\epsilon+\eta)\}$$

with boundary

$$\partial_\pm U = \{x : Q(x) = \pm\epsilon, \|x_\mp\|^2 = \eta\}$$
$$\partial_0 U = \{x : \|x_-\|^2 \|x_+\|^2 = \eta(\epsilon+\eta)\}$$

That this is indeed the boundary can be seen by looking at the function $\theta\colon x \mapsto (\|x_-\|, \|x_+\|) \in \mathbb{R}^2$

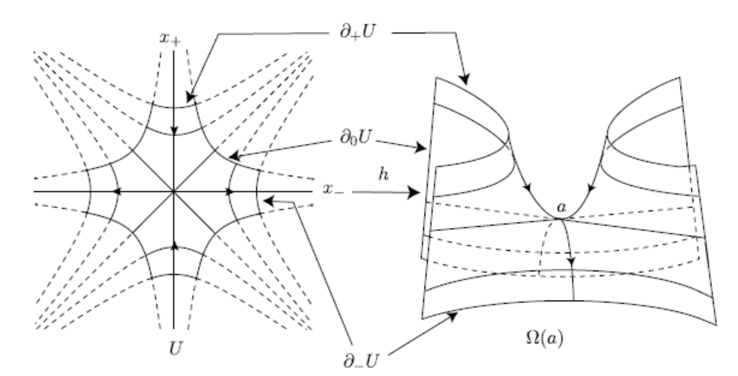

Figure: The standard ball, h is the diffomorphism from the Morse chart to the neighbourhood $\Omega(a)$ around critical point a. Figure taken from [2].

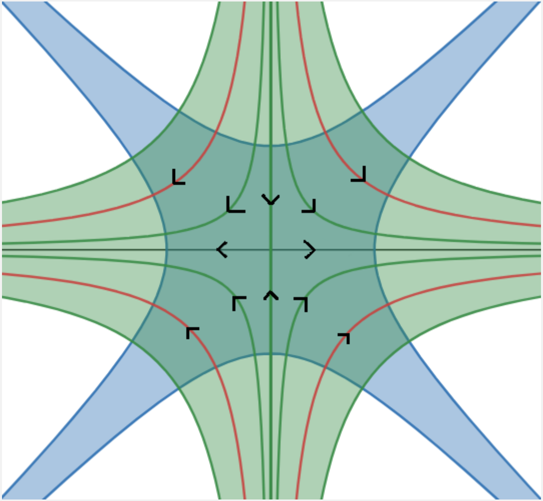

Figure: Here we see the regions in the standard ball and the flow lines are marked with arrows. Figure made using Desmos [4].

Flows

With the above description of the flow in local coordinates and from the diagrams, intuitively if a flow line enters a Morse chart and leaves, then it cannot re-enter. More precisely, since any flow line entering and leaving U must do so through $\partial_\pm U$ and since f is strictly decreasing, the flow line cannot re-enter.

Therefore, on a compact manifold, if a trajectory doesn't terminate in critical points, then beyond some time the value of $df(X)$ is strictly less than some constant $\delta < 0$. This leads to a contradiction as f cannot keep decreasing on a compact manifold.

Theorem

Let M be compact and γ a trajectory for X, then there exist critical points c, d of f such that $\lim_{t\to\infty} \gamma(t) = c, \lim_{t\to-\infty} \gamma(t) = d$.

Stable and unstable manifolds

Since M is compact, X generates a one-parameter family of diffeomorphisms $\phi_t \colon M \to M$ corresponding to the flows along X. Since all trajectories end at/start from critical points, we define

$$W^s(a) = \{x \in M | \lim_{t\to\infty} \phi_t(x) = a\},$$

$$W^u(a) = \{x \in M | \lim_{t\to-\infty} \phi_t(x) = a\}.$$

$W^s(a)$ is called the stable manifold of a and $W^u(a)$ is called the unstable manifold. There is a similar notion in the context of dynamical systems where we consider the iterates of some homeomorphism $M \to M$.

Stable and unstable manifolds

We claim that the stable and unstable manifolds are in fact embedded submanifolds of M. Recall the local flow equation (1)

$$\gamma : t \mapsto (c^1 e^{2t}, \dots, c^k e^{2t}, c^{k+1} e^{-2t}, \dots, c^n e^{-2t})$$

In a chart as above, the elements in $W^s(p)$ have $x_- = 0$ and all must pass through a sphere S^{n-k-1} of the form $\|x_+\|^2 = \epsilon$. So, we consider the map

$$\Phi\colon S^{n-k-1} \times \mathbb{R} \to M$$
$$(x, t) \mapsto \phi_t(0, x)$$

This map lands in $W^s(p) \setminus \{p\}$. If $\Phi(z_1, t_1) = \Phi(z_2, t_2)$, then $\phi_{t_1 - t_2}(0, z_1) = (0, z_2)$ which forces $(z_1, t_1) = (z_2, t_2)$ as z_1, z_2 are on the sphere and the flow is radially inward. Therefore Φ is injective.

Stable and unstable manifolds

For $(z_0, t_0) \in S^{n-k-1} \times \mathbb{R}$, under Φ, the pushforward of $\partial/\partial t$ is $X(\phi_{t_0}(0, z_0)) = \phi_{t_0 *}(X(0, z_0))$ and the pushforward of $T_{(0,z_0)}S^{n-k-1}$ is $\phi_{t_0 *} T_{(0,z_0)}S^{n-k-1}$. Since ϕ_{t_0} is a diffeomorphism and $X(0, z_0) \notin T_{(0,z_0)}S^{n-k-1}$, we see that $\Phi_*(z_0, t_0)$ is injective. Intuitively, a linear independence of X and vectors tangential to the sphere at $t = 0$ is preserved at all times.

Lastly, observe that on a Morse chart as above, the stable set is an embedded submanifold, i.e., Φ is a local embedding onto its image near $t = 0$. The translations $l_T : t \mapsto T + t$ and $x \mapsto \phi_T(x)$ are diffeomorphisms and we have

$$\begin{tikzcd} S^{n-k-1} \times \mathbb{R} \arrow[r, "\Phi"] \arrow[d, "1\times l_T"'] & M \arrow[d, "\phi_T"] \\ S^{n-k-1} \times \mathbb{R} \arrow[r, "\Phi"] & M \end{tikzcd}$$

Stable and unstable manifolds

Given $z_0 \in S^{n-k-1}$ there is a neighbourhood $(z_0, 0) \in V \subset S^{n-k-1} \times \mathbb{R}$ and a neighbourhood W in M such that $\Phi\colon V \to W \cap W^s(p)$ is a homeomorphism. Since $\phi_T(W^s(p)) = W^s(p)$, we restrict the diagram above to get

$$\begin{array}{ccc} V & \xrightarrow{\Phi} & W \cap W^s(p) \\ {\scriptstyle 1\times l_T}\downarrow & & \downarrow{\scriptstyle \phi_T} \\ V' & \xrightarrow{\Phi} & \phi_T(W) \cap W^s(p) \end{array}$$

Three of these are homeomorphisms, so the bottom map is also a homeomorphism. So, $W^s(p) \setminus \{p\}$ is an embedded submanifold. Using the diffeomorphism $(0,1) \to \mathbb{R} : s \mapsto \ln(s/1-s)$, we get the embedding $\Psi\colon S^{n-k-1} \times (0,1) \to M$. In local coordinates

$$\Psi\colon (x_{k+1}, \dots, x_n, s) \mapsto (0, \dots, 0, x_{k+1}((1-s)/s)^2, \dots, x_n((1-s)/s)^2)$$

Stable and unstable manifolds

It is easy to see that Ψ extends to $S^{n-k-1} \times (0, 1]$ and from there to a disc. It follows that $W^s(p)$ is a submanifold of M of codimension $\mathrm{Ind}(p)$ and is diffeomorphic to a disc. A similar analysis applies for $W^u(p)$ where we just have to change a few signs (or consider $-f$). Thus, the stable and unstable manifolds are in fact manifolds and

$$\dim W^u(p) = \operatorname{codim} W^s(p) = \mathrm{Ind}(p).$$

Another description of p is as a point of compactifying $W^s(p) \setminus \{p\}$ as stated in [2].

Height function on Torus

Let T denote a torus embedded in $\mathbb{R}^3$ with a parametrization given by

$$(u, v) \mapsto (r \cos u \cos v + R \cos v, r \cos u \sin v + R \sin v, r \sin u)$$

where $u, v \in [0, 2\pi]$ (closed interval to cover the torus), $R > r > 0$. We take a height function given by the projection onto the x-axis and an easy calculation shows that the critical points of this function are given when (u, v) is one of $(0, 0), (\pi, 0), (0, \pi), (\pi, \pi)$.
One also finds that the indices are 2(max), $1, 1, 0$(min) respectively.
Label the critical points A, B, C, D as in the figure. The stable and unstable manifolds are depicted in the figure below. The flow lines move from one critical point to another.

Height function on Torus

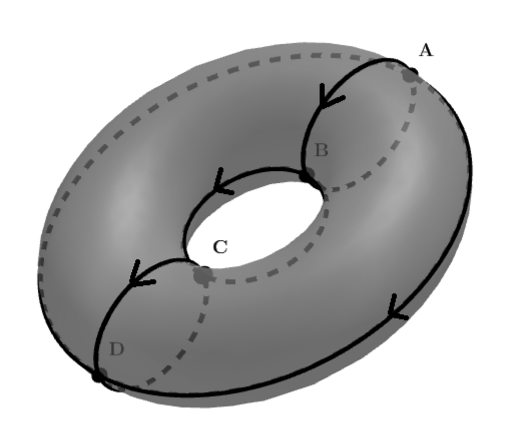

Figure: Flow lines on the torus viewed obliquely. Figure made using Geogebra [4].

Height function on Torus

At A, we have a 2-dimensional unstable manifold, consisting of flows going downwards. At B we have a one dimensional stable manifold consisting of flows originating from A and a one dimensional unstable manifold. Other points are similar. The exact relation and number of flows between critical points leads to Morse homology.

The Morse Theorems

Theorem

Let f be a smooth real valued function on a manifold M (not necessarily compact; without boundary). Let $a < b$ and suppose $f^{-1}[a, b]$ is compact and without critical points of f. Then $M^a = f^{-1}(-\infty, a]$ is diffeomorphic to and a deformation retract of M^b and furthermore, the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.

Theorem

Let p be a non-degenerate critical point with index k. Let $f(p) = c$ and suppose $f^{-1}[c - \epsilon, c + \epsilon]$ is compact and without any other critical point for some $\epsilon > 0$. Then for all sufficiently small ϵ, the set $M^{c+\epsilon}$ has the same homotopy type as $M^{c-\epsilon}$ with a k-cell attached.

The reader is referred to [1], [2] for proofs of these theorems.

Reeb's theorem

Theorem

Let M be a compact manifold (without boundary) admitting a Morse function f with exactly 2 critical points, then M is homeomorphic to the sphere.

Proof.

Being compact M admits a maximum and minimum and these are critical points of index $n, 0$ respectively. Since connected components are closed, hence compact, we conclude that M is connected. By composing with a diffeomorphism, we may assume $f(M) = [0, 1]$. By Morse lemma, for sufficiently small $\epsilon > 0$ the sets $f^{-1}[0, \epsilon], f^{-1}[1 - \epsilon, 1]$ are discs of dimension n.
By Morse theorem, $M^{1-\epsilon}$ is diffeomorphic to M^{ϵ}, therefore, $M = M^{1-\epsilon} \cup f^{-1}[1 - \epsilon, 1]$ is a union of two discs glued along the boundary $f^{-1}(1 - \epsilon)$. It is a standard result that this is homeomorphic to S^n. ∎

Decomposition of the Torus

We return to the height function on the torus. Here we have 4 critical points of index $0, 1, 1, 2$. So, by Morse theorems, we get the following decomposition of the torus

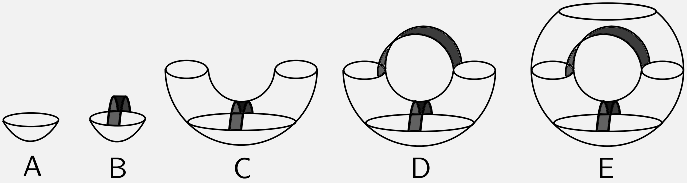

Figure: Decomposition of the torus

where A is a disc (homotopic to a point), and from A to B we add a 1 cell which results in a half-torus C (obtained by stretching the handle upwards). To C we add another 1-cell resulting in D which is homotopic to E. Finally to E we add a 2-cell and "close" the torus.

Morse Inequalities

Although we have discussed Morse theorems, the original formulation as developed by Marston Morse was quite different. In fact, the original theory involved what are now called Morse inequalities which presented bounds on the Betti numbers (ranks of the homology groups) in terms of the number of critical points of a given index. Let f be a Morse function on a compact manifold M and let C_k denote the number of critical points of index k. Let R_k denote the kth Betti number of M, i.e., the rank of $H_k(M)$. Then Morse inequalities say [1]

$$R_k \leq C_k,$$

$$\sum(-1)^k R_k \leq \sum(-1)^k C_k,$$

$$R_k - R_{k-1} + \cdots \pm R_0 \leq C_k - C_{k-1} + \cdots \pm C_0.$$

Note that the first two follow from the third. These are powerful inequalities and help in the computation of homology of spaces such as the complex projective space.

Further topics

We talked about the stable and unstable manifold and an important condition is the Smale condition which is when all the stable and unstable manifolds intersect transversally. This condition then lets us develop Morse homology which is essentially done by counting the number of trajectories from one critical point to another. Furthermore, we have only talked about manifolds without boundary, but Morse theory can be developed in the context of manifolds with boundary as well. These ideas are also involved in h-cobordism and the generalized Poincare conjecture.

References and further reading

- Milnor, *Morse Theory*
- Audin and Damian, *Morse Theory and Floer Homology*
- Lee, *Riemannian Manifolds: an Introduction to curvature*
- Graphing calculator Desmos
  Graphing calculator Geogebra