Baire Category Theorem and some applications to Functional Analysis

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Abstract

The Baire Category Theorem is a fundamental result in general topology and functional analysis that give sufficient conditions for a space to be a Baire space. In this talk we are going to define what a Baire space is and provide proofs of the Baire Category Theorems. Then we apply these theorems to prove certain important results from Functional Analysis, namely the Open Mapping Theorem, Closed Graph Theorem and the Bounded Inverse Theorem, the latter two as applications of the Open Mapping Theorem. Some background in topology and functional analysis is assumed.

Definition

A topological space X is said to be a Baire space if for every countable collection $\{A_n\}_{n\geq 1}$ of closed sets with empty interior, their union $\cup_{n\geq 1} A_n$ also has empty interior in X. Equivalently, for every countable collection $\{U_n\}_{n\geq 1}$ of open dense sets, their intersection $\cap_{n\geq 1} U_n$ is also dense in X.

Exercise Let A be a subset of a topological space X, then A has empty interior if and only if its complement is dense in X.

Examples The space of rationals is not a Baire space because the singletons are closed with empty interior, but their union $\mathbb{Q}$ doesn't have empty interior. The space of integers, or any space with discrete topology, is vacuously Baire because there is no nonempty set with empty interior.

Originally, Baire defined a subset A of a space X to be of the *first category* if it was contained in a countable union of closed sets having empty interiors; otherwise, it was said to be in the *second category*. Using this terminology, a space is Baire if and only if every nonempty open set is in the second category. Sets in the first category also go by the name of *meagre* sets.

The notion of being Baire is somewhat similar to how in measure theory the countable union of null sets is null, which follows from the countable (sub)additivity of measures, in that a countable union of "negligible" sets is again "negligible".

Lemma

An open subset Y of a Baire space X is also a Baire space.

Sketch.

One can show that if A is a closed subset in Y with empty interior (in Y), then its closure $\overline{A}$ in X also has empty interior. Given this, if $\{A_n\}$ is a countable collection of closed sets in Y with empty interior, then we look at the collection $\{\overline{A_n}\}$. Since X is Baire, their union has empty interior.

Now if U is open in Y and contained in $\cup A_n$, then U is open in X (because Y is an open subset of X) and contained in $\cup A_n \subset \cup\overline{A_n}$ which forces U to be empty, thus completing the proof. ∎

The Baire Category Theorems

Theorem

(Baire Category Theorem 1) A complete pseudometric space is a Baire space.

Theorem

(Baire Category Theorem 2) A locally compact Hausdorff space is a Baire space.

Psuedometric space

Definition

A pseudometric on a set X *is a function* $d\colon X \times X \to \mathbb{R}_{\geq 0}$ *such that*

1. $d(x,x) = 0 \forall x \in X$.
2. $d(x,y) = d(y,x) \forall x, y \in X$.
3. $d(x,z) \leq d(x,y) + d(y,z) \forall x, y, z \in X$.

We can talk of pseudometric spaces, sequences in such spaces and their metric completions just the way one speaks of in metric spaces. The only difference between a pseudometric and a metric is that the "distance" between two different points under a pseudometric can be zero, i.e., the space need not be T_0.

A pseudometric space which is not a metric space

Consider the space C of functions on $[0,1]$ which are square integrable. This space is equipped with a "norm"

$$\|f\| = \sqrt{\int_{[0,1]} f^2 dt}.$$

This induces a pseudometric

$$d(f,g) = \sqrt{\int_{[0,1]} (f-g)^2 dt}$$

which is not a metric because functions which are nonzero, but vanish almost everywhere have distance 0 from the zero function. In order to obtain a metric space structure, we quotient C by the subspace of functions which vanish everywhere to obtain the space $l^2([0,1])$.

Proof of BCT1

Theorem

(Baire Category Theorem 1) A complete pseudometric space is a Baire space.

Proof

Let $\{A_n\}$ be a countable collection of closed sets with empty interiors. To show that $\cup A_n$ has empty interior, given any open set U we must find an element of U not contained in any of the A_ns.
Let U be an open set.
Since A_1 has empty interior, there is a point $x_1 \in U \setminus A_1$, hence an $\epsilon_1 > 0$ such that $x_1 \in B(x_1, \epsilon_1) \subseteq U \setminus A_1$. Set $U_1 = B(x_1, \epsilon_1/4)$ and note that $\overline{U_1} \subset U \setminus A_1$.
Now, A_2 has empty interior, so there is a point $x_2 \in U_1 \setminus A_2$, hence an $\epsilon_2 > 0$ such that $x_2 \in B(x_2, \epsilon_2) \subseteq U_1 \setminus A_2$. Set $U_2 = B(x_2, \epsilon_2/4)$. Note that $x_2 \notin A_1 \cup A_2, \epsilon_2 < \epsilon_1/2$ and $\overline{U_2} \subset U_1 \setminus A_2$.

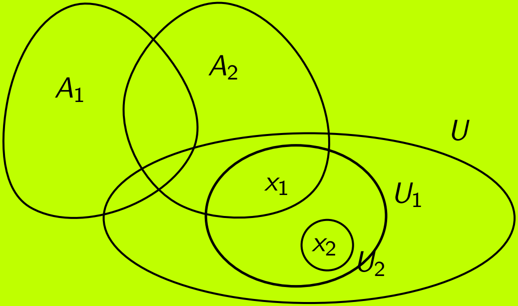

Figure: BCT 1

Continuing this way, we obtain $x_n \in U_{n-1} \setminus A_n$ and hence an $\epsilon_n > 0$ such that $x_n \in B(x_n, \epsilon_n) \subseteq U_{n-1} \setminus A_n$, and we set $U_n = B(x_n, \epsilon_n/4)$. Then $x_n \notin A_1 \cup \cdots \cup A_n, \epsilon_n < \epsilon_1/2^{n-1}$ and $\overline{U_n} \subset U_{n-1} \setminus A_n$.

The diameter of U_n is less than $\epsilon_1/2^{n-1}$ and $\overline{U_n}$ contains the tail $x_n, x_{n+1}, \ldots$. The sequence $\{x_n\}_{n \geq 1}$ is Cauchy and hence converges to some x. Since $\overline{U_n}$ is closed, $x \in \overline{U_n} \subset U_{n-1} \setminus A_n$, hence $x \notin A_n$. Therefore, $x \in U$ is an element not in the union $\cup A_n$ as required, hence X is Baire. ∎

Proof of BCT2

Proof.

Let X be locally compact Hausdorff. Given a closed set A and $x \notin A$, there is a neighbourhood U of x such that $\overline{U}$ is compact and disjoint from A. Let $\{A_n\}$ be a countable collection of closed sets with empty interior and let U be any open set.

Since A_1 has empty interior, there is an $x_1 \in U \setminus A_1$, hence an open set U_1 such that $\overline{U_1} \cap A_1 = \emptyset, \overline{U_1} \subseteq U$ and $\overline{U_1}$ is compact. Now, having obtained $\overline{U_{n-1}}$ we choose a point in $U_{n-1} \setminus A_n$ and then a neighbourhood U_n of this point so that

$$\overline{U_n} \cap A_n = \emptyset \text{ and } \overline{U_n} \subset U_{n-1}.$$

We then have a descending chain $\overline{U_1} \supset \overline{U_2} \supset \cdots \supset \overline{U_n} \supset \ldots$ and since $\overline{U_1}$ is compact, the intersection $\cap \overline{U_n}$ is nonempty. Let $x \in \cap \overline{U_n}$, then $x \notin \cup A_n$, hence X is a Baire space. ∎

Open Mapping Theorem

Theorem

Let X, Y be Banach spaces and $A\colon X \to Y$ be a continuous surjective linear map, then A is an open map.

One can weaken the hypotheses to X being an F-space, Y a topological vector space, A being continuous and linear such that $A(X)$ is in the second category. For a proof of this stronger version, see [3]. An F-space is a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a metric d such that with respect to the topology induced by d, addition and scaling are continuous, d is translation invariant (i.e., $d(x+a, y+a) = d(x, y)$) and the space is complete.

Proof

Suppose U is open in X and let $A(x) \in A(U)$. Consider the translate $U - x$ which is an open neighbourhood of 0 in X and its image, by linearity, is $A(U) - A(x)$ which contains the origin. If we show that $A(U) - A(x)$ contains a neighbourhood of $0 \in Y$, then by translating back, we know that $A(U)$ contains a neighbourhood of $A(x)$. Since A is surjective, this shows that $A(U)$ is an open set. Therefore, it suffices to consider open neighbourhoods of $0 \in X$, in particular it suffices to show that the image of open balls centred at the origin contains a neighbourhood around $0 \in Y$.

To this end, let $U = B_X(0,1), V = B_Y(0,1)$ be unit balls around the origin in X, Y respectively. By surjectivity, the closed sets $\{\overline{A(kU)}\}_{k\geq 1}$ cover Y, and by Baire Category Theorem, some $\overline{A(kU)}$ has nonempty interior.
Let us say $B_Y(c,r) \subset \overline{A(kU)}$ for some $c \in Y$ in which case, for every $v \in V, c + rv \in \overline{A(kU)}$. Using linearity, it follows that

$$rv \in \overline{A(kU)} + \overline{A(kU)} \subseteq \overline{A(2kU)}$$

where the containment follows from the fact that any $x \in \overline{A(kU)}$ is the limit of a sequence in $A(kU)$.
Applying linearity once again, we obtain $V \subseteq \overline{A(lU)}$ where $l = 2k/r$. What this means is that given $v \in V$ and $\epsilon > 0$, there is an $x \in lU$, i.e., $x \in X$ with $\|x\|_X < l$ such that $\|v - Ax\| < \epsilon$. We claim that $V \subseteq A(2lU)$, and then shrinking the sets would complete the proof.

One idea is to take a sequence x_n such that $A(x_n)$ converge to v, and hope that the x_n converge to some x, and comparing the norms gives us the required containement, however there is no reason for the x_n to be a convergent sequence.
We instead approximate v as a sum. First observe that $V/n \subset \overline{A(l/nU)}$. First, obtain x_1 with $\|x_1\|_X < l$ such that

$$\|v - Ax_1\|_Y < 1/2.$$

Now, $v - Ax_1$ is an element closer to 0 than v. Obtain an $x_2 \in X$ with $\|x_2\|_X < l/2$ such that

$$\|v - Ax_1 - Ax_2\|_Y < 1/4.$$

Continuing this way, having obtained $x_1, \ldots, x_{n-1}$, obtain an $x_n \in X$ with $\|x_n\|_X < l/2^{n-1}$ such that

$$\|v - A(x_1 + \cdots + x_n)\|_Y < 1/2^n.$$

It is clear that $\sum x_n$ is defined in X, let it be x and by continuity of $A, Ax = v$. Moreover, $\|x\|_X < 2l$ as required. Therefore, $V \subseteq A(2lU)$. Therefore, $A(U)$ contains the ball $V/2l$. Hence A is an open map. ■

If N denotes the kernel of A, then we have a vector space isomorphism $A \colon X/N \to Y$. Give X/N the quotient topology (via the quotient map $x \sim y \iff x - y \in N$), then this map is continuous. Since A is open, this resulting quotient map is also open, therefore, $A \colon X/N \to Y$ is a linear homeomorphism between vector spaces.

In the finite dimensional case

Suppose X, Y were finite dimensional above, then A induces a linear isomorphism $\bar{A}\colon X/N \to Y$ as usual. Now however, because $X/N, Y$ are finite dimensional, this linear isomorphism is a homeomorphism (the inverse is given by another matrix), hence open. Since the projection $\pi\colon X \to X/N$ is open (Exercise), it follows that $A = \bar{A} \circ \pi$ is an open map.
Why does this proof fail in general? The answer is that in the infinite dimensional case, the invertibility of the linear transformation doesn't guarantee us that the inverse is continuous, that is the essence of the Bounded Inverse Theorem which we shall get into shortly and the proof will use the Open Mapping Theorem.

Closed Graph Theorem

There are two theorems that go by the name "Closed Graph Theorem". Given a map $f: X \to Y$ between two objects, the graph of f is the subset $\Gamma f = \{(x, f(x)) : x \in X\} \subseteq X \times Y$.

Theorem

(Topological) Let $f: X \to Y$ be a map between two spaces and let Y be compact Hausdorff, then f is continuous if and only if its graph is closed (in $X \times Y$).

Theorem

(Functional analysis) Let $f: X \to Y$ be a linear map between two Banach spaces, then f is continuous if and only if it's graph is closed (in $X \times Y$).

Proof

Suppose f is continuous and $(x, y) \notin \Gamma f$, so $y \neq f(x)$. Since Y is Hausdorff, obtain $y \in U, V \ni f(x)$ disjoint open. Using continuity of f, we have an open W such that $x \in W \subset f^{-1}(V)$. It then follows that $(x, y) \in W \times U \cap \Gamma f = \emptyset$. Note that we only needed the fact that Y was Hausdorff.

Conversely, assume that the graph is closed. Now $X \times Y$ is a Banach space with the pythagorean (or any equivalent) norm say. Moreover, it is complete because a sequence is convergent/Cauchy iff its components are. Because f is linear and Γf is closed, it is also a Banach space with the restricted norm.

The projection $\pi_1 \colon \Gamma f \to X$ is a continuous, linear bijection, hence by the open mapping theorem it is an open map, i.e., $\pi_1^{-1} \colon X \to \Gamma f$ is a continuous map. Composing this with the projection $\pi_2 \colon X \times Y \to Y$ to the second coordinate, we see that $f = \pi_2 \circ \pi_1^{-1} \colon X \to Y$ is continuous as required. ■

Remark. As in the case of the open mapping theorem, the closed graph theorem also holds more generally in the context of F-spaces. In fact, the proof above holds without change in the case of F-spaces.

Bounded Inverse Theorem

Theorem

Let X, Y be a Banach space, $T\colon X \to Y$ a bounded linear map. If T is invertible, then T^{-1} is bounded.

Proof.

This is a direct consequence of the open mapping theorem. Since T is open, T^{-1} is continuous, hence bounded. ∎

Lemma

A linear map $T\colon X \to Y$ is bounded if and only if it is continuous.

Proof.

Since we are dealing with metric spaces, continuity is equivalent to sequential continuity and by linearity it suffices to check continuity at $0 \in X$. If T is continuous at 0, then there is a $\delta > 0$ such that for $\|x\|_X < \delta, \|Tx\|_Y < 1$. Then given any $x \in X$, we have

$$\left\| T\left(\frac{\delta x}{\|x\|_X}\right) \right\|_Y = \frac{\delta}{\|x\|_X} \|Tx\|_Y \leq 1$$

hence $\|Tx\|_Y \leq (1/\delta) \|x\|_X$.

Conversely, suppose T is bounded, say $\|Tx\|_Y \leq C \|x\|_X$. Then given $\epsilon > 0$, for $\|x\|_X < \epsilon/C$ we have $\|Tx\|_Y < \epsilon$, hence T is continuous at 0. ∎

Uniform Boundedness Principle

Theorem

Let X be a Banach space, Y a normed linear space and suppose $\{T_n\}$ is a family of continuous linear transformations from X to Y such that for each $x \in X, \{\|T_n x\|\}$ is bounded, then $\{\|T_n\|\}$ is bounded.

Proof

For each integer k, let $X_k = \{x : \sup_n \|T_n(x)\| \leq k\}$, then each X_k is closed as if $x \notin X_k$, then $\|T_m(x)\| > k$ for some m which means that in some neighbourhood U of $x, \|T_m(y)\| > k \forall y \in U$, therefore $U \subseteq X_k^c$. By assumption, $\cup X_k = X$, so by Baire category theorem some X_k has nonempty interior, say X_N.

So, there is some $x_0 \in X, \epsilon > 0$ such that $\overline{B_X(x_0, \epsilon)} \subseteq X_N$. Now, given $u \in X$ with $\|u\| \leq 1$, we have for any $T = T_l$,

$$\begin{aligned} \|Tu\| &= \epsilon^{-1} \|T(x_0 + \epsilon u) - T(x_0)\| \\ &\leq \epsilon^{-1}(\|T(x_0 + \epsilon u)\| + \|T(x_0)\|) \\ &\leq \epsilon^{-1}(N + N) = \epsilon^{-1}(2N) \end{aligned}$$

Therefore, each T_l is bounded by the same constant. ■

Some concluding remarks

There are some more results that are consequences of the Baire category theorem, the following theorem for example.

Theorem

Let X be a space, (Y, d) a metric space. Let $f_n \colon X \to Y$ be a sequence of continuous functions converging pointwise to a function $f \colon X \to Y$. If X is a Baire space, the set of continuity points of f is dense in X.

In our proof of BCT, we had to invoke some form of the Axiom of Choice. In fact, the first theorem is equivalent over ZF to the Axiom of Dependent Choice. However, a restricted form of the theorem in which the complete metric space is also assumed to be separable is provable in ZF without choice. For more about this, see [4].

With regards to our proofs of the theorems in Functional Analysis, we required the open mapping theorem first. However, the three theorems are actually equivalent (under reasonable assumptions), see [5] and [6].

Extras - locally compact T_2 spaces

Let X be locally compact Hausdorff and let a closed set A and point $x \notin A$ be given. Because A is closed we obtain an open neighbourhood U of x disjoint from A. By the local compactness we have an open neighbourhood W of x such that $\overline{W}$ is compact. Now, $\overline{W} \cap A$ is a closed subset of a compact set, hence itself compact, and it is disjoint from x, so we can find an open neighbourhood V of x such that $\overline{V}$ is disjoint from $\overline{W} \cap A$. Consider the neighbourhood $U_1 = U \cap W \cap V$. Now this is disjoint from A, and it's closure is a closed subset of W, hence compact and moreover,

$$\overline{U_1} \cap A \subseteq \overline{W \cap V} \cap A \subseteq \overline{V} \cap \overline{W} \cap A = \emptyset$$

which completes the proof.

Extras II - topological CGT

Suppose Γf is closed in $X \times Y$. First we show that the projection $\pi \colon X \times Y \to X$ is a closed map. Let C be closed in $X \times Y$ and suppose $(x, y) \notin C$. Then for each $y \in Y$ there is a neighbourhood $U_y \times V_y$ containing (x, y) disjoint from C.
Using compactness of Y we obtain a finite subcover from the V_ys, and hence a neighbourhood U of X such that $U \times Y$ is disjoint from C, then U is disjoint from $\pi(C)$. Next, given a closed set $C \subseteq Y$ we know that $\Gamma f \cap X \times C$ is closed in $X \times Y$ and it's projection to X is precisely $f^{-1}(C)$ which is closed. This means that f is continuous. Note that we only need Y to be compact.

References and further reading

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