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How To Train Your Jordan Curve

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December 2023
IMS 2023, BITS Hyderabad

Abstract

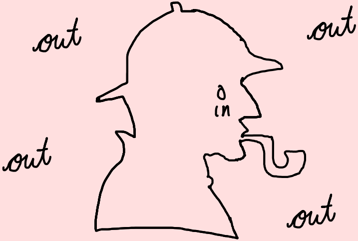
The Jordan curve theorem says that a simple closed curve in the plane divides it into two parts - the inside and the outside. While the statement is fairly intuitive, the proof is far from trivial. There have been multiple proofs of the theorem over the last several decades, including a few formal proofs.
The paper in question attempts a proof using finite polygonal covers of arcs and curves and proves Jordan-like results for these covers by first simplifying the covers so the intersection of polygons is easy to handle.
In my talk I will motivate some of the reasoning behind these results and provide some interesting counterintuitive examples in the realm of curves in the plane. Time permitting, we will get into the details of the proofs.

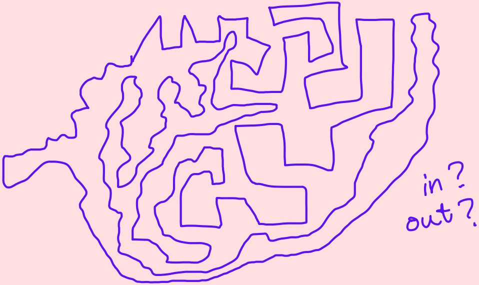
Statement of Jordan Curve Theorem

A Jordan curve is an injective continuous map $J\colon S^1 \to \mathbb{R}^2$. An arc shall mean an injective map $[0,1] \to \mathbb{R}^2$.

Theorem (Jordan Curve Theorem (JCT))

The complement $\mathbb{R}^2 \setminus J$ of a Jordan curve J has two components, one bounded and another unbounded with J as the common boundary.





A brief history

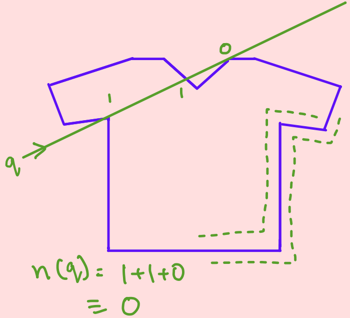
While the statement is deceptively simple, it was Bolzano who first conjectured the precise statement and observed that it requires a proof.

The first proof was provided by Jordan, hence the name, but it was assumed to be flawed. Later, Veblen provided the first rigorous proof. Since then, there have been multiple proofs using a variety of techniques like algebraic topology, analysis, nonstandard analysis, constructive mathematics, graph theory and so on. Thomas Hales and others have even compiled a formal proof.

Easy case: Polygons

One defines a parity function n on the complement $\mathbb{R}^2 \setminus P$ of a polygon P counting the mod 2 number of times a ray to infinity crosses P. Because n is continuous and surjective, $\mathbb{R}^2 \setminus P$ has at least two components.

Using some analysis akin to a tubular neighbourhood of the polygon, one proves that there are exactly two components and that they both have P as the boundary.



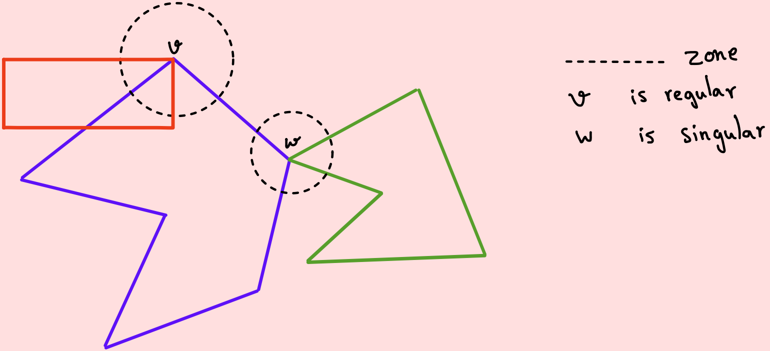
How to proceed?

Having proved the case for polygons, some of the earlier proofs proceed by approximating arbitrary Jordan curves by polygons. At the time of writing the proofs, I wasn't comfortable with such results, so I tried to prove it differently.
Similar to the earlier proofs, one could start by hoping that you can cover the curve J with polygons so that the "outer boundary" of the cover will point out what the inside and outside of J are.

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a collection of polygons in the plane. Let $V(\mathcal{P})$ denote the set of all intersections of edges from P_is (this includes vertices).

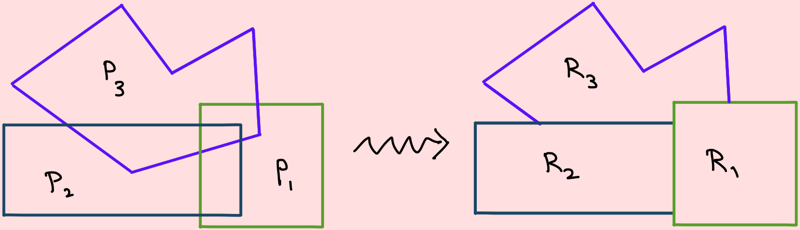
Some definitions

For each $v \in V(\mathcal{P})$ define a zone to be a ball not containing any other point of $V(\mathcal{P})$. This ball has radii coming from edges of $\mathcal{P}$. Call v regular if every component outside $\mathcal{P}$ has at most one sector in a zone around v. Else v is singular.



Refining polygons

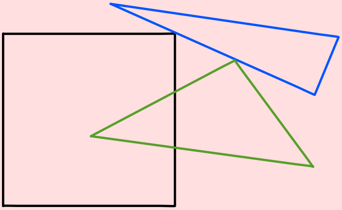
A first result is that we can rewrite the union $\cup \overline{i(P_j)}$ where $i(P)$ is the inside of the polygon P, as a union $\cup \overline{i(R_k)}$ such that the polygons R_k only intersect along the edges.
Proof idea: We remove the parts of P_j that are inside P_k for $k < j$ and repeats this process. Care has to be taken in showing that the resulting shapes are still polygons and so on.



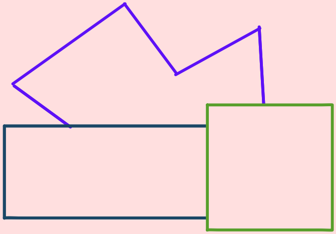
All vertices regular before & after

Importantly, the set of vertices doesn't change. If anything, it becomes smaller. Thus, if the original collection of polygons was regular, then so is the new one.

We'll say a collection of polygons $\mathcal{P}$ has shallow intersections, if the polygons only intersect along edges.



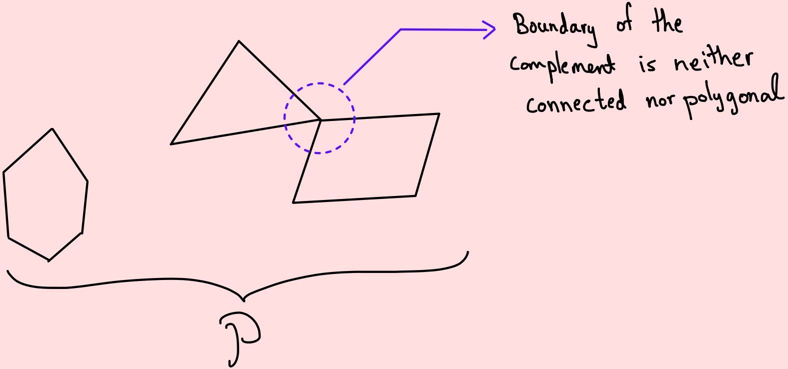
no shallow intersections

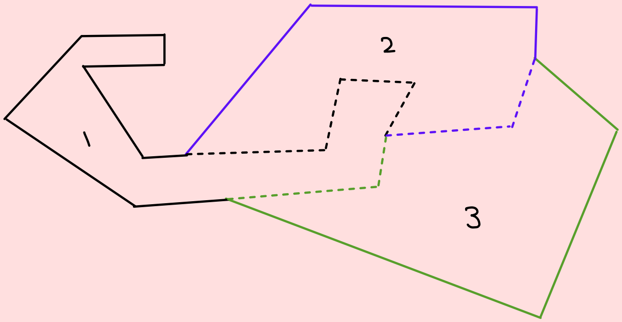


shallow intersections

What do the components outside look like?

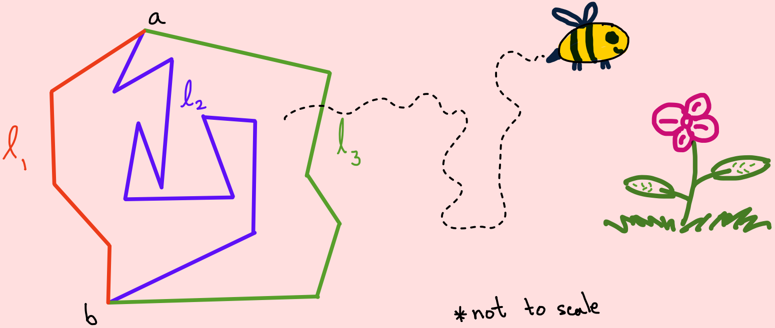
Our second result is that if $\mathcal{P}$ is a regular collection of polygons with shallow intersections and a connected intersection graph, then all components in the complement of $\mathcal{P}$ have polygonal boundaries with vertices from $V(\mathcal{P})$.





The unbounded complement has polygonal boundary given by the solid edges.

Suppose we have three polygonal paths l_1, l_2, l_3 between points a, b. Taking two paths at a time, we have three polygons. Intuitively, a path from the inside of say $l_2 \cup l_3$ to the total outside not touching $i(l_1 \cup l_2)$ should intersect l_3. This result is used in the proof quite a few times to show that certain paths should intersect.



Jordan-like result II

Suppose $\mathcal{P}$ is a regular collection of polygons with shallow intersections. Let R be a component outside $\mathcal{P}$ whose boundary is a polygon. If there are paths outisde R joining "opposite" edges/vertices, then the paths should intersect.

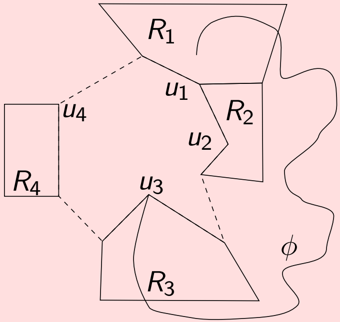
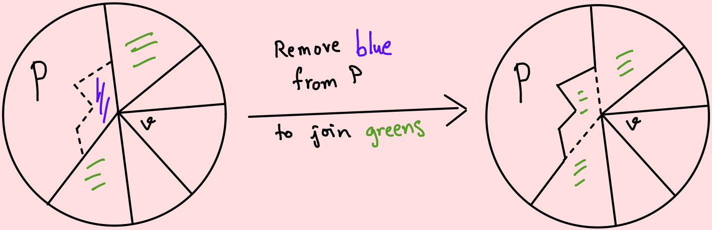


Figure: Any path from R_2 to R_4 should intersect ϕ.

Chisels, files and sandpaper

Fix a, b in the plane and let ϕ_1, ϕ_2 be two arcs between them. We can cover ϕ_1 by polygons in a number of ways. Having obtained a cover, to apply the previous results, we'd like to remove singular vertices and make it so that the polygons have shallow intersection. We needn't worry about the intersection graph because the Jordan curve makes it connected.

To remove singular vertices, we remove "triangles" from problematic sectors in a zone:



We then make sure that if two edges intersect, then ϕ_1 must pass through the intersection. The algorithm is similar to removing singular vertices in that we remove a rectangular neighbourhood of problematic intersections from each of the polygons (a similar thing can be done when the edges aren't parallel):

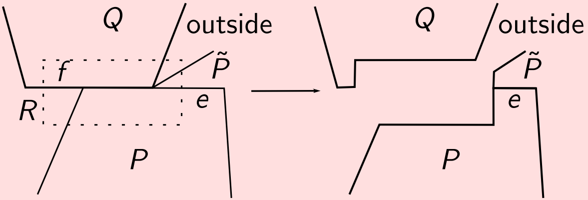


Figure: Removing a problematic intersection

Special covering

With the tools of the previous slides, we can refine a cover of ϕ_1 to obtain, what the paper calls, a special cover. This is a cover of ϕ_1 by polygons $\mathcal{P} = \{P_1, \ldots, P_n\}$ such that

- $\mathcal{P}$ is regular, with shallow intersections
- a, b are contained within two polygons T_1, T_2 that don't intersect
- If ϕ_2 intersects $\overline{i(P_j)}$, then P_j is T_1 or T_2
- If two polygons meet, then ϕ_1 passes through each of their intersections.

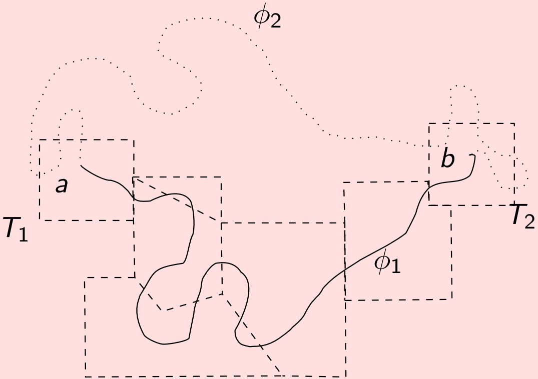


Figure: Example of a special covering

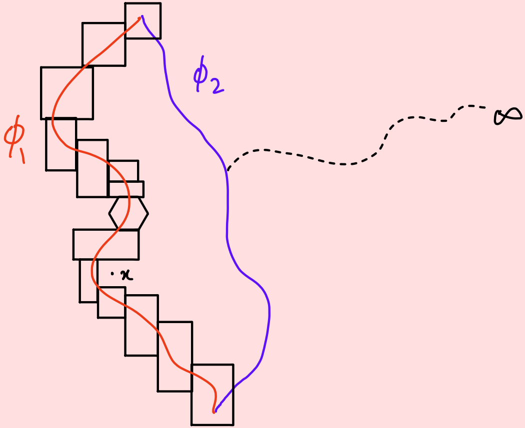
Consequence of the special covering - in words

Theorem

Suppose ϕ_1, ϕ_2 are two arcs meeting only at the ends a, b. Let B be a circle or a polygon such that $\phi_1 \cup \phi_2 \subset i(B)$. Suppose there is a point $c \in \phi_2 \setminus \{a, b\}$ with a path ϕ going from c to a point outside B such that $\phi \cap \phi_1 = \emptyset$. Then given any $x \notin \phi_1 \cup \phi_2$, there is a polygonal cover $\mathcal{P}$ (with each polygon inside B) of ϕ_1 such that x is in the unbounded component of $\mathbb{R}^2 \setminus \mathcal{P}$.

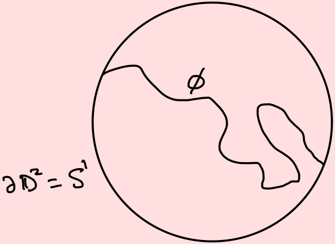
Consequence of the special covering - user friendly

The theorem states that if we have ϕ_1, ϕ_2 as pictured and a path ϕ from a point in ϕ_2 going off to infinity, then given any point $x \notin \phi_1 \cup \phi_2$, we can cover ϕ_1 by polygons that avoid x



Jordan Curve Theorem

The paper then goes on to prove first that an arc inside the unit disc going from a point on S^1 to another separates the disc into two parts. This is very much like the true Jordan curve theorem, with the only difference is that one half of the Jordan curve is a circular arc, hence very much under our control.
Having proved this, the paper bounds an arbitrary Jordan curve inside a circle and splits it into two by drawing a chord. Using similar techniques, the paper proves the Jordan Curve Theorem.



Arc in a disc separates the disc

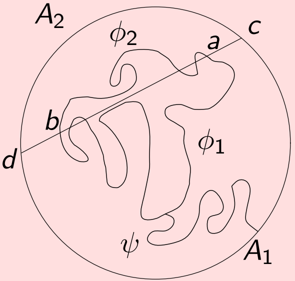
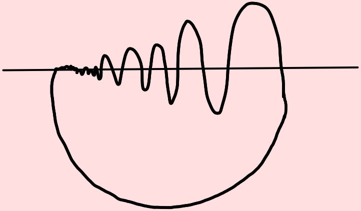


Figure: Jordan Curve inside a disc

A wintry morning at a frozen lake

What goes wrong with extending the parity function approach to arbitrary curves? Firstly, there's no need for a ray to intersect the curve a finite number of times. An immediate example is something like the Warsaw circle where the x-axis intersects something like $\sin(1/x)$ infinitely many times. This example leads us to think that we could shift our ray a little but and obtain a finite number of intersections, but then we have things like the Koch snowflake, named after Helge von Koch.



Warsaw Circle

Koch snowflake

In the
limit, we have a Jordan curve which intersects (horizontal) lines infinitely many times (or doesn't intersect at all).

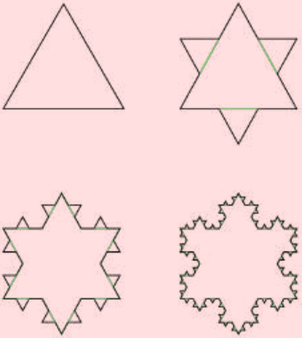


Figure: The first four stages in constructing the Koch snowflake

Osgood Curve

Ok, so passing a ray doesn't let us know whether the starting point is inside or outside, how about claiming that alternate "gaps" (the curve divides the ray into gaps) on the ray are from different components. Intuitively, this feels right...but there's a problem with even defining what "alternating gaps" are.

If I shouldn't be able to find the next gap, that means there should be an infinite number of gaps between any two gaps, this again points to a fractal like nature. This led me to a very interesting object: the Osgood curve, discovered by William Fogg Osgood.

This is a non self intersecting curve (hence, not space filling) with positive measure in $\mathbb{R}^2$. In particular, it is a Jordan curve and goes against our intuitions. There's no way to tell two gaps on a ray are adjacent. Interestingly, there's an Osgood curve of measure λ for any $\lambda \in (0, 1)$.

Figure: Osgood arc

Wada Lakes

Another interesting intuition-defying example is when we consider the boundary of curves. It seems intuitive that if I have a curve that separates the plane, then the curve should be the common boundary of all components. Because there are only "two sides" to the curve, there can't be more than two components.

But this is not the case. To be in the boundary, we need neighbourhoods of points on the curve to intersect all the components in the complement. What if we could have components interleaving near the curve, but staying on "one side"? Then we could have more than two components.

These are the Lakes of Wada, named after Takeo Wada. These are three disjoint regions in the plane all of which have the same boundary. So it's not obvious that the complement of a Jordan curve should have two components.

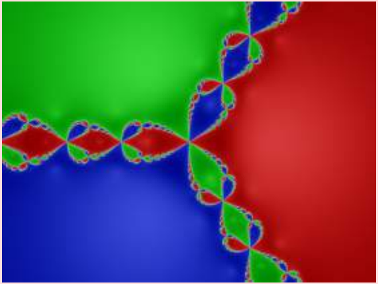


Figure: Wada Lakes

Thank you

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References and further reading III

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