Solutions to Graph theory notes

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Notations and terminology

For vertices u, v in a graph G a $u - v$ path refers to a sequence of distinct vertices $w_0 w_1 \dots w_n$ such that $w_0 = u, w_n = v$ and each (w_i, w_{i+1}) is an edge in G. For a $u - v$ path P and a $v - w$ path Q, the path $R = P + Q$ refers to the walk obtained by first going along P and then along Q. This may not always be a path.

I have used both $\setminus$ and $-$ for indicating set differences, the latter used mostly in the context of removing sets of edges or vertices from graphs.

Indicator function for a set A is denoted be $\mathbf{1}[x \in A]$.

$\subset$ shall mean a proper subset and $\subseteq$ is used to indicate subsets in general.

Chapter 3

Exercise 3.1.6

Let $G = (V, E)$ be a graph. Let $V = \{v_1, \dots, v_n\}$.
Take $S = E$ and $S_i = \{e \in E \colon v_i \in e\}$ for $1 \leq i \leq n$.
Now let G' be the insertion graph obtained from S. Clearly there is a bijection between $V(G)$ and $V(G')$ by identifying S_i with v_i.
Now suppose for vertices x and y in $V, x \sim y \Rightarrow S_x \cap S_y \neq \emptyset$. Therefore, $S_x \sim S_y$ in G'.
Conversely, suppose $S_x \sim S_y$ in G'. Then, by the definition of S_x, S_y, x and y are adjacent in G.
Hence, $G \cong G'$. Thus, every graph can be obtained as an intersection graph.

Exercise 3.1.7

Take $H = \mathbb{Z}^d$. It has generators $S = \{e_i, -e_i \colon 1 \leq i \leq d\}$. The Cayley graph on $(H, +)$ is precisely the Euclidean lattice. Therefore, the Euclidean lattices are Cayley graphs.

Exercise 3.1.12

Done by inspection.

Exercise 3.1.13

Yes, by inspection.

Exercise 3.1.14

Let $G = (V, E)$ be a graph on n vertices and e edges. For a vertex $v \in G$, let $d(v)$ denote its degree and for an edge $e = (v, w)$, let $d(e)$ denote its degree in the line graph $L(G)$ of G. We know

$$\sum_{v \in V} d(v) = 2e$$

Also, for an edge $e = (v, w), d(e) = d(v) + d(w) - 2$. Thus, for a vertex $v, w \in G$ with $w \sim v$

$$\begin{aligned}
d(v,w) &= d(v) + d(w) - 2 \\
\Rightarrow \sum_{w \sim v} d(v,w) &= d(v)^2 + \sum_{w \sim v} d(w) - 2d(v) \\
\Rightarrow \sum_{v \in V} \sum_{w \sim v} d(v,w) &= \sum_{v \in V} d(v)^2 + \sum_{v \in V} \sum_{w \sim v} d(w) - 4e
\end{aligned}$$

Now the left hand side sums the degree of every edge twice, therefore it is $2 \sum_{e \in E} d(e)$. In the second term on the right the degree of v occurs $d(v)$ times. So,

$$2 \sum_{e \in E} d(e) = 2 \sum_{v \in V} d(v)^2 - 4e$$

Let e' denote the number of edges in $L(G)$. Then,

$$4e' = 2 \sum_{v \in V} d(v)^2 - 2 \sum_{v \in V} d(v)$$

$$e' = \sum_{v \in V} \frac{d(v)(d(v)-1)}{2} = \sum_{v \in V} \binom{d(v)}{2}$$

Exercise 3.2.2

Yes; edges are mapped to edges, so the composition does the same.

Exercise 3.2.3

$\Rightarrow$ Suppose H is an induced subgraph, then for every $v, w \in V(H), (v, w) \in E(G) \Rightarrow (v, w) \in E(H)$. So, clearly H is the maximal subgraph on $V(H)$.
$\Leftarrow$ Let H be the maximal subgraph on $V(H)$. If H is not an induced subgraph then $\exists\ v, w \in V(H)$ such that (v, w) is an edge in G but not in H. Adding this edge to H creates a graph larger than H contradicting maximality of H. Therefore H is an induced subgraph on $V(H)$.

Exercise 3.2.5
Suppose G is bipartite and $V(G) = V_1 \sqcup V_2$. Let K_2 have vertices $\{1,2\}$.Then, consider the map

$$\begin{aligned} \phi\colon G &\to K_2 \\ x &\mapsto i \text{ if } x \in V_i \end{aligned}$$

Clearly ϕ is a homomorphism. Conversly, suppose

$$\phi\colon G \to K_2$$

is a graph homomorphism. Take $V_1 = \phi^{-1}(1)$ and $V_2 = \phi^{-1}(2)$, then $V(G) = V_1 \sqcup V_2$ and the only edges of G are those with one end in V_1 and the other in V_2 as K_2 doesn't have loops. Hence, G is bipartite.

Exercise 3.2.6
If $\phi\colon G \to K_k$ is a homomorphism, then upon defining $V_i = \phi^{-1}(i)$ for $i \in \{1, 2, \dots, n\}$ we have $V(G) = \sqcup_{i=1}^{i=n} V_i$ and similar to Exercise 3.2.5, G is k-partite.
Conversely, suppose G is k-partite, $V(G) = \sqcup_{i=1}^{i=n} V_i$. Then, consider the map

$$\begin{aligned} \phi\colon G &\to K_k \\ x &\mapsto i \text{ if } x \in V_i \end{aligned}$$

Clearly ϕ is a graph homomorphism.

Exercise 3.2.7
Each injective homomorphism corresponds to such a tuple and each such tuple gives rise to an injective homomorphism. Note that since G is labelled, we need to consider all such permutations.

Exercise 3.2.9

- Complete graph K_n is $(n-1)$-reguar. So, average degree $a(K_n) = (n-1)$. Since the graph is complete any bijection is an automorphism, so $Aut(K_n) = S_n$.
- We can't say anything about intersection graphs or Delaunay graphs without any information on the graphs.
- Cayley graphs are regular with degree same as the cardinality of set of generators.

Exercise 3.2.10

- In the complementary graph G^c of $G = (V, E)$,
  $d(v)_{G^c} = n - d(v)_G$ so,
  Minimum degree $= n - \Delta(G)$
  Maximum degree $= n - \delta(G)$
  $a(G^c) = \sum_{v \in V} \frac{n - d(v)}{n}$
- In the line graph $L(G)$ of $G = (V, E)$,
  $d(v, w) = d(v) + d(w) - 2$, so
  Minimum degree $= \min_{(v,w) \in E}(d(v) + d(w) - 2)$
  Maximum degree $= \max_{(v,w) \in E}(d(v) + d(w) - 2)$
  From Exercise 3.1.14, $|E(L(G))| = \sum_{v \in V} \binom{d(v)}{2}$, so
  $a(L(G)) = \frac{\sum_{v \in V} \binom{d(v)}{2}}{|E(G)|}$

Exercise 3.2.12
Yes. If we start from a vertex and its three neightbours, by 3-regularity each of the three neighbours must be connected to exactly 2 of the remaining vertices. Now there are 6 edges left to be distributed among 6 vertices which corresponds to a cycle of length 6. Now backtracking we start with a cycle of length 6. Since antipodes of a 6 cycle are not adjacent, they must have a common neighbour. This adds three vertices of degree 2. The remaining vertex must be connected to each of these three vertices. So, there is only one such graph. Since the Petersen graph satisfies these properties, every such graph is isomorphic to the Petersen graph.

Exercise 3.2.14
Let $G = (V, E)$ be a graph, $v, w \in V$. Suppose $P = v_0 v_1 \dots v_n$ is a walk in G with $v_0 = v$ and $v_n = w$.
Induction on n.
If $n = 1$, we are done.

Assume that if there is a walk from v to w of length less than n there is a self avoiding walk from v to w.
If P is a self avoiding walk we are done. Suppose P is not self avoiding, then some vertex x is repeated, $P = v_0v_1 \dots (v_k = x) \dots (v_l = x) \dots v_n$ with $k < l$. Consider the new walk $P' = v_0v_1 \dots v_kv_{l+1} \dots v_n$. P' is a walk from v to w of length less than n. So by induction hypothesis, there is a self avoiding walk from v to w. By induction hypothesis, if there is a walk from v to w then there is a self avoiding walk from v to w. Since v, w were arbitrary, this is true for every pair of vertices.
Conversely, if there is a self avoiding walk from v to w then it is a walk from v to w.
The case for paths is now clear.

Exercise 3.2.15

- Reflexivity : $v \to v$ by a constant path
- Symmetry : $v \to w$ then there is a path $P = (v = v_0)v_1 \dots (v_n = w)$ from v to w then $P' = v_nv_{n-1} \dots v_1v_0$ is a path from w to v.
- Transitivity : $u \to v \to w$ then there are paths $P = (u = v_0)v_1 \dots (v_n = v)$ and $Q = (v = w_0)w_1 \dots (w_m = w)$ from u to v and v to w respectively. Then $R = P + Q$ is a walk from u to w. By Exercise 3.2.14 there is a path from u to w, so $u \to w$.

Thus, $\to$ is an equivalence relation.
Let C_v be the induced subgraph of the component containing v and let H be the maximal connected graph containing v.
Since there is a path from v to every vertex w in C_v, $V(C_v) \subseteq V(H)$.
Conversely, since H is connected, there is a path from v to every vertex w in H, $V(H) \subseteq V(C_v)$.
Again by maximality of H, C_v being an induced subgraph, it follows that $C_v = H$.

Exercise 3.2.16

This is obvious.

Exercise 3.2.17

Let $G_1 \cong G_2, v_i$ be a vertex of minimum degree in G_i and w_i be a vertex of maximum degree in G_i and let $\phi\colon G_1 \to G_2$ be an isomorphism. For any vertex v of G_1 we have $d(v) = d(\phi(v))$.

- $\delta(G)$: We have $d(v_1) = d(\phi(v_1)) \Rightarrow \delta(G_2) \le d(v_1)$. Similarly $d(v_2) = d(\phi^{-1}(v_2)) \Rightarrow \delta(G_1) \le d(v_2) \le d(v_1)$.Therefore $d(v_1) = d(v_2) \Rightarrow \delta(G_1) = \delta(G_2)$.
- $\Delta(G)$: This is similar to $\delta(G)$.
- $d(G)$: Follows from $d(v) = d(\phi(v)) \;\forall\; v \in V(G_1)$ and ϕ being an isomorphism.
- $\beta_0(G)$: This follows from the fact that if $P = v_0v_1 \dots v_n$ is a path in G_1 from v_0 to v_n then $\phi(P) = \phi(v_0)\phi(v_1) \dots \phi(v_n)$ is a path in G_2 from $\phi(v_0)$ to $\phi(v_n)$ and the fact that ϕ is an isomorphism.

Exercise 3.2.18

Let G have $n \ge 2$ vertices. Since G is simple a vertex v can have degree $0, 1, \dots, n-1$.
Suppose there are k vertices with degree 0.
If $k \ge 2$, we are done.
If $k = 0$, then every vertex has degree $1, 2, \dots, n-1$. By pigeon hole principle, two vertices must have the same degree.
If $k = 1$, then there is no vertex of degree $(n-1)$. So every other vertex must have degree $1, 2, \dots n-2$. Again, by pigeon hole principle, among the $(n-1)$ vertices with nonzero degree there must be two vertices of the same degree.

Exercise 3.3.1

Let $G = (V, E)$ be a connected graph, $, x, y, z \in V$.
If $x = y$, then clearly $d(x, y) = 0$. Conversely, if $d(x, y) = 0$ then the smallest weight path between x and y must have length 0 as otherwise every edge in such a path adds a positive weight to the distance function making $d(x, y) \ne 0$.
If P is a smallest weight path from x to y then it is also a smallest weight path from y to x as reversing a path gives a path in the other direction. Therfore, $d(x, y) = d(y, x)$.
If P is a smallest weight path from x to y and Q is a smallest weight path from y to z then $P + Q$ is a walk from x to z.
Let R be a smallest weight path from x to z then $w(R) \le w(P + Q) = w(P) + w(Q)$ as the shortest weight walk from x to z is either the underlying path in $P + Q$ or is smaller than that.
By the points mentioned above, it is clear that d is a metric on V.

Exercise 3.3.2

Let G be the Cayley graph of the free group generated by $S = \{s_1, s_2, \dots, s_n, -s_1, -s_2, \dots, -s_n\}$. Let $k = |S|$.
Let $g = g_0g_1 \dots g_n$ be a word such that $g_ig_{i+1} \ne e \;\forall\, 0 \le i \le n{-}1$ and $g_i \in S \;\forall\, i$. Then $P = (e)(g_0)(g_0g_1) \dots (g_0g_1 \dots g_{n-1})(g)$ is the shortest path from e to g as the group is free and if there were a shorter path we would have a non trivial relation

in the free group. So, $d(e, g) = n + 1$.
For the same reasons G has no cycles. So, there are exactly $k(k-1)^n$ words at a distance of $n+1$ from e.
Therefore, $|B_n(e)| = 1 + \sum_{i=1}^{i=n} k(k-1)^{(i-1)}$.
In fact the converse is also true as then there would be $k(k-1)^n$ words at a distance of $n+1$ from e and there are precisely those many possible words of length $n+1$ in a free group with a symmetric set of generators of cardinality $k \Rightarrow$ there are no non trivial relations satisfied by the generators.

Exercise 3.3.3

Firstly, the number of ways to distribute n alike things into r distinct groups is $\binom{n+r-1}{r-1}$.
We shall use the convention $\binom{n}{k} = 0$ when $n < k$ and $\binom{0}{0} = 1$.
A point $(z_1, z_2, \ldots, z_d)$ in $\mathbb{Z}^d$ has a distance $|z_1| + |z_2| + \cdots + |z_d|$ from the origin O.
So there are $\binom{n+d-1}{d-1}$ points at a distance n from O with each $z_i \geq 0$.
Now the sign of each such coordinate can be flipped to get a point in the other orthants and each such point can be obtained by flipping signs in such a manner. However, those coordinates that are zero pose a problem.
If exactly k coordinates are zero then there are $\binom{d}{k}\binom{n-d+k+d-k-1}{d-k-1} = \binom{d}{k}\binom{n-1}{d-k-1}$ coordinates in the positive orthant at a distance n from O. We obtain this number by choosing k coordinates, setting them to zero and adding a 1 in the remaining coordinates.
Now $k \in 0, 1, \ldots, d-1$. So if $f(n)$ represents the number of point at a distance $n, n \geq 1$ from the origin, then

$$f(n) = \sum_{k=0}^{k=d-1} 2^{d-k}\binom{d}{k}\binom{n-1}{d-k-1}$$

as the sign of each nonzero coordinate can be flipped to get a new point.
Then,

$$|B_n(0)| = 1 + \sum_{i=1}^{i=n} f(n) = 1 + \sum_{i=1}^{i=n}\sum_{k=0}^{k=d-1} 2^{d-k}\binom{d}{k}\binom{i-1}{d-k-1}$$

.

Exercise 3.3.7

The Euclidean lattice is invariant under translation homomorphisms. So $\pi_n(0) = \pi_n(x)$ for every vertex x.
Now, a path of length $n+m$ is a path of length n followed by one of length m. So,

$$\pi_{n+m}(0) \leq \pi_n(0)\pi_m(0)$$

as every path of length n can be extended from the end vertex in at most $\pi_m(0)$ ways, i.e. we have an injection from {self avoiding walks of length $n+m$ from origin} $\rightarrow$ {self avoiding walks of length n from origin} $\times$ {self avoiding walks of length m from origin}.
Thus,

$$\ln(\pi_{n+m}(0)) \leq \ln(\pi_n(0)) + \ln(\pi_m(0))$$

and hence, $\ln(\pi_n(0))$ is subadditive.
By Fakete's Lemma, $\{\frac{\ln(\pi_n(0))}{n}\}_{n\geq 1}$ converges. Since the function e^x on $\mathbb{R}$ is continuous the statement follows.

Exercise 3.4.3

Let A be the adjacency matrix of the graph. We know that $A^k_{(i,j)}$ is the number of walks from i to j of length k, so the required sum is $Tr(A^l)$.
Also, for a matrix M with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m, Tr(M) = \sum_{i=1}^{i=m} \lambda_i$.
Now, since A is symmetric, the eigenvalues of A^k are $\lambda_1^k, \ldots, \lambda_n^k \; \forall \; k \in \mathbb{N}$.
Therefore, $Tr(A^l) = \sum_{i=1}^{i=n} \lambda_i^l$.

Exercise 3.4.4

Let M is the $n \times n$ matrix with all entries 1 and A is the adjacency matrix of K_n, then $A = M - I$. The eigenvalues of M are n and 0. Suppose λ is an eigenvalue of A and X is a corresponding eigenvector. Then,

$$AX = (M - I)X = MX - X$$

$$\Rightarrow \lambda X = MX - X$$

$$\Rightarrow MX = (\lambda + 1)X$$

$$\Rightarrow \lambda + 1 = n \text{ or } 0$$

$$\Rightarrow \lambda = n-1 \text{ or } -1$$

Further, by comparing algebraic multiplicities, multiplicity of $(n-1)$ is 1 and -1 is $n-1$. By Exercise 3.4.3, the number of closed walks of length k is $(n-1)^k + n - 1$ if k is even and $(n-1)^k - n + 1$ if k is odd.

Exercise 3.5.2

Let G be a finite graph.
$\Rightarrow$ Suppose G is Eulerian. Let $C = v_0v_1\dots v_k$ be an Eulerian circuit. Then clearly G is connected as for any two vertices, there is a subsequence in C that gives a walk between them.

Claim : Every closed circuit is the union of edge disjoint cycles.

Proof : Let C be a circuit in G with k edges. Induction on k.
$k = 1$ implies C is a single loop, so the result is true.
Assume that claim holds for all circuits of length less than k. If C is a cycle, we are done. Else some vertex v is repeated, say $C = vv_1v_2\dots(v_p = v)v_{p+1}\dots(v_k = v)$ with $1 < p < k$. Then $C_1 = vv_1\dots(v_p = v)$ and $C_2 = vv_{p+1}\dots(v_k = v)$ are two edge disjoint closed circuits of length less than k. So, by induction hypothesis, they are union of edge disjoint cycles. As a consequence C is the union of edge disjoint cycles (observe that C_1 and C_2 are edge disjoint).
Hence, the claim is true by induction. ■

Since C is a closed circuit, it follows that C is a union of edge disjoint cycles.
$\Leftarrow$Suppose G is connected and a union of edge disjoint cycles $G = C_1 \cup C_2 \cup \dots \cup C_k, k \geq 1$. Since G is connected, each of these cycles must have a vertex in common with some other cycle.
Induction on k.
$k = 1, G$ is a cycle and we are done as cycles are Eulerian.
Assume the statement holds for every union of edge disjoint cycles with less than k cycles. Choose C_1. Form a new graph H by removing the edges of C_1 from G. Each component of H is now a union of edge disjoint cycles as C_1 was edge disjoint from the other cycles. By induction hypothesis, each component of H is Eulerian.
Let $C_1 = v_0v_1\dots v_m$. Start at v_0 and proceed along the edges of C_1. Everytime we encounter a vertex in some component of H, go along the Euler circuit of that component and come back to the mentioned vertex. Since G is connected every component in H must have some vertex in C_1 . Thus, we have traversed every edge of H once and every edge of C_1 once and ended at v_0, i.e. we have an Euler circuit of G. Therefore G is Eulerian.
The proof is completed by induction.

Exercise 3.5.4

$\Rightarrow$ Suppose G is an Eulerian multigraph and $C = v_0v_1\dots v_k$ is an Eulerian circuit. Then for a vertex v of G, every edge of v is counted exactly once and every occurence of v in C increases degree of v by 2, hence its degree is even. This is true for every vertex of G.
$\Leftarrow$Induction on number of edges e of G.
$e = 2$, then $n = 1$ or 2 as on three vertices we need at least 3 edges for every vertex to have degree ≥ 2 (G is connected and sum of degrees is $2e$).
Case 1: $n = 1$, then G is two loops, and clearly Eulerian.
Case 2: $n = 2$, then G must have two edges between two vertices and hence is clearly Eulerian.
Assume that the theorem holds for such graphs with less than e edges. If G has a loop l, then remove one of them to get a connected graph with every vertex having even degree with $e-1$ edges. By induction hypothesis, $G - l$ is Euelrian, and adding l back to G keeps it Eulerian.
So assume G has no loops. If G has two edges e_1, e_2 between some vertices v, w then remove both and induction hypothesis applies. $G \setminus \{e_1, e_2\}$ has an Eulerian circuit C that starts and ends at v then $e_1 + e_2 + C$ is an Eulerian circuit of G.
So now suppose G has no loops or parallel edges. Then G is simple and this case has been dealt with in the original theorem.
Thus, the proof is completed by induction.

Exercise 3.5.5

Let $W = w_0w_1\dots w_kw_0, k$ even, be a closed walk of odd length in G. Induction on k.
$k = 0$, then W is a loop and hence a cycle of odd length.
$k = 2$, then W must have three vertices as it is impossible to have a closed walk of length 3 on 2 vertices as any graph on 2 vertices is bipartite. Thus W is a cycle.
Assume that the statement holds for closed odd walks of length less than k. If W is a cycle we are done, else some vertex v is repeated. Assume, without loss of generality, $v = w_0$, then $W = w_0w_1\dots(w_p = w_0)w_{p+1}\dots w_kw_0, p < k$. Then $W_1 = w_0w_1\dots w_p$ and $W_2 = w_0w_{p+1}\dots w_kw_0$ are two closed walks of length less than k. Since W has odd length, one

of these must have odd length and hence, by induction hypothesis, must contain an odd cycle. This odd cycle is an odd cycle in W as well.
Thus, the statement is true by induction.

Chapter 4

Exerise 4.1.2

Claim : A connected graph on n vertices has at least $(n-1)$ edges.

Proof : Induction on n. If $n = 2$, then claim is true.
Assume that the claim is true for any connected graph on $n-1$ vertices. If every vertex in G has degree greater than 2 then, from

$$2e = \sum_{v \in V(G)} d(v) \geq \sum_{v \in V(G)} 2$$

it follows that $e \geq n-1$ where e is the number of edges in G.
Now suppose $v \in V(G)$ has degree 1, then $G - v$ is a connected graph on $n-1$ vertices so has at least $n-2$ edges. These together with the one edge removed give G at least $n-1$ edges.
So, the claim is true by induction. ■

Lemma : A connected graph on n vertices is a tree iff it has $n-1$ edges.

Proof : Suppose G has $n-1$ edges. If G has a cycle C and e is an edge in C then $G - e$ is connected as if e occurs in a path P, we can always replace e with $C - e$ to get a path in $G - e$. However, $G - e$ has $n-2$ edges contrtadicting the above lemma.Thus, G has no cycles and hence is a tree.
Conversely, suppose G is a tree. If every vertex has degree ≥ 2 then G has a cycle contradicting the assumption that G is a tree. So, there is at least one vertex v of degree 1. $G - v$ is also a tree. Induction on n.
$n = 1$, G is a single vertex and hence has no edges. Assume the statement to be true for all trees on $n-1$ vertices. Then using the notation above $G - v$ has $n-2$ edges $\Rightarrow$ G has $n-1$ edges.
Hence, the lemma is true by induction. ■

(1) $\Rightarrow$ (2) : Each component is a tree. Adding the edges component wise, G must have $n - \beta_0(G)$ edges.
(2) $\Rightarrow$ (1) : Let $T_1, T_2, \ldots, T_k$ be the components of G with $n_1, n_2, \ldots, n_k$ vertices.Then T_i has at least $n_i - 1$ edges. Since G has $n-k$ edges, it follows that each T_i has $n_i - 1$ edges, i.e. each T_i is a tree. Hence G is a forest and is acyclic.

Exercise 4.1.3

We know that a tree G on n vertices has $n-1$ edges.
Case 1: Every vertex of G has degree ≥ 2. Then $2e = \sum_v d(v) \geq \sum_v 2 = 2n \Rightarrow e \geq n$, a contradiction.

Case 2: Exactly one vertex w of degree 1. Then $2e = \sum_v d(v) \geq 1 + \sum_{v \neq w} 2 = 2n - 1 \Rightarrow e \geq n - \frac{1}{2}$, a contradiction.

Therefore, there must be at least two vertices of degree 1.
Observe that there are trees with exactly 2 vertices of degree 1, eg. paths on n vertices. In fact, one can show that path graphs are the only trees with exactly 2 vertices of degree 1.

Exercise 4.1.6

Fix a degree sequence $(d_1, d_2, \ldots, d_n)$, two of which must be one. We can still assign Prufer codes to these trees and the mapping will be injective. The vertex v_i shall appear $d_i - 1$ times. Since Prufer codes uniquely give back the tree, we just need the number of Prufer codes where i appears $d_i - 1$ times and this is precisely $\binom{n-2}{d_1-1,\ldots,d_n-1}$ proving the tree counting theorem. When we add this number over all degree sequences, we get $(1 + 1 + \cdots + 1)^{n-2} = n^{n-2}$ giving back Cayley's formula.

Exercise 4.1.8

(1) $\Rightarrow$ (2) : If $T \subseteq G$ is a spanning forest then each component of T is a tree, hence removing any edge of T adds more components as any connected graph must have at least $n-1$ edges where n is the number of vertices and trees have exactly those many edges.
(2) $\Rightarrow$ (3) : If T contains a cycle C in a component T' of T and e is an edge in C then $T' - e$ is connected as given vertices u, v in T' if the $u - v$ path in T' contains e we can replace e by all edges of $C - e$ giving a $u - v$ walk in $T' - e$. So $T - e$ has as many components as T contradicting (2). Therefore, T doesn't contain cycles.
T has as many components as G, so if (u, v) is an edge in G but not in T then u and v must be in one component T' of T as otherwise they would be in different components of G. So, T' contains a $u - v$ path thus adding edge (u, v) creates a cycle in T.
(3) $\Rightarrow$ (1) : If G contains an isolated vertex not in T then adding it to T creates a larger graph, so we may assume T

contains all isolated vertices. Adding an edge (u,v) creates a cycle, so T must contain u and v. Thus, T spans G. Since T is acyclic it is a spanning forest.
Let $C_1, C_2, \ldots, C_k$ be the component of G and $T_{11}, T_{12}, \ldots, T_{1n_1}, T_{21}, \ldots, T_{2n_2}, \ldots T_{k1}, T_{k2}, \ldots, T_{kn_k}$ be the components of T with T_{ij} in C_i for $1 \leq i \leq k$. Suppose $n_1 > 1$ then T_{11} and T_{12} are disjoint. For vertices u in T_{11} and v in T_{12} there is a $u-v$ path P in C_1 adding any edge of P to T creates a cycle, so there has to be a $u-v$ path in T a contradiction. So each $n_i = 1$ for $1 \leq i \leq k$. Thus T has k components.

Exercise 4.1.9

If e is an edge between two different components of H then adding it clearly reduces $\beta_0(H)$ by 1. Else, e is an edge between vertices in the same components of H, say $e = (u,v)$. Since they are in the same component, H has a simple $u-v$ path P and this doesn't contain e. So, $P+e$ is a cycle C with $C-e = P$ not a cycle in H.

Exercise 4.1.11

If e is in some cycle of G then as done in Exercise 4.1.8 removing e doesn't disonnect the component containing that cycle and as a result doesn't increase the number of components of G. So, e is not a cut edge.
Conversely, if $e = (u,v)$ is a not a cut edge then there is a $u-v$ path in $G-e$ and they remain in the same component after removing e. It is then clear that e is part of some cycle in G.

Exercise 4.1.14

Fix an $e = (u,v)$ in $T - T'$ then there is a $u-v$ path $P \neq e$ in T'. $T-e$ has two components with u and v in different compoenents. Now P is a path from u to v, so some edge f of P must go from one component to another. $f \neq e$ as P doesn't contain e. Then $T-e+f$ is connected with $n-1$ vertices and spans G which has n vertices. So $T-e+f$ is a tree.

Exercise 4.2.3

Let C be a cycle, $e = (u,v)$ an edge of maximum weight in C and M an MST of G. If $e \notin M$ we are done, so assume $e \in M$. $M-e$ has two components with u and v in different compoenents. C is a cycle, so some edge f of C, $f \neq e$ must go from one component to another. Then $M-e+f$ is a tree with $w(M-e+f) \leq w(M)$ as e is an edge with maximum weight in C. Since M is an MST, we must have $w(f) = w(e)$. So, if e is the only maximum weight edge in C we have a contradiction. Since $f \notin M, M$ doesn't contain an edge of maximum weight from C, proving the theorem.

Exercise 4.3.3

Let M, M' be two minimal spanning trees. Now given an edge $e = (u,v) \in M \setminus M'$, $M'+e$ has a cycle $C = v_0 v_1 \ldots v_t$ with $v_0 = u$ and $v_t = v$. $M-e$ has two components G_1, G_2 containing u, v respectively. Now since M is spanning, every $v_i \in M, 0 \leq i \leq t$. If $v_1 \in G_2$ then (v_0, v_1) is an edge from G_1 to G_2, else $V_1 \in G_1$ and we look at v_2. If $v_2 \in G_2$, we have an edge in C which goes from G_1 to G_2, else $v_2 \in G_1$. Continuing this way, we get an edge going from G_1 to G_2, say f. Note that f must be in M' for otherwise, we would have a cycle in M. Now $M-e+f, M'+e-f$ are both spanning trees and by the condition of minimality, we conclude that $w(e) = w(f)$. We can do the same for every edge of $M \Delta M'$. So, M and M' have the same weights occuring, say $w_1 < w_2 < \cdots < w_k$.
Now the idea is to insert edges from M to M' one by one and change M' to M. Let $\{e_1, e_2, \ldots, e_p, g_1, g_2, \ldots, g_r\}$ be edges in M of weight w_1 and $\{f_1, f_2, \ldots, f_q, g_1, g_2, \ldots g_r\}$ be edges in M' of weight w_1. Suppose $p > q$. Then, $M'+e_1$ has a cycle and every edge in this cycle must have weight w_1, and at least one edge is from M'. So $\exists\, f_{i_1}$ such that $M'+e_1-f_{i_1}$ is a spanning tree. Next add e_2 and remove a different edge f_{i_2}. Continuing this way, once we remove each f_i, if e_{q+1} exists then upon adding it to the latest M', we get a cycle all of whose edges are from M which is a contradiction. So, $p = q$, i.e. M and M' have the same number of edges of weight w_1. At this stage the edges in M' of weight w_1 are $\{e_1, e_2, \ldots, e_p, g_1, g_2, \ldots, g_r\}$. We can repeat the same process with the larger weights. Suppose we have shown that the first s weights occur the same number of times, then if the next weight occurs more in one tree than the other, say more in M' then, we transfer edges from M to M' of smaller weights by the above process and repeat the above process for this larger weight. So, the result holds by induction.
Thus the two minimal spanning trees have the same multiset of weights.

Exercise 4.3.4

A minimal spanning forest of a weighted graph G is a union of minimal spanning trees of each components.

Exercise 4.3.5

Fix a vertex u, we will induct on the length of the minimum weight path from x to u. If it has length 1, then it is clear that $d(x,u) = d_G(x,u)$. Suppose the length is k and assume that the statement holds for all paths of length less than k. Let the minimum weight path from x to u be $P = v_0 v_1 \ldots v_{k+1}$ with $v_0 = x, v_{k+1} = u$ and let $v_k = v$ then $Q = v_0 \ldots v_k$ is the minimum weight path from x to v. So, by hypothesis, $d(x,v) = d_G(x,v)$. Now because of the algorithm, since v has a shorter path, v was considered before u and at that stage $t(u)$ changed to $w(P)$ and after that it could have only decreased but that is not possible since P is the minimum weight path from x to u. Therefore $d(x, u) = d_G(x, u)\ \forall\ u \in G$.

Chapter 5

Exercise 5.1.5

Using the notation used in the proof of Mantel's theorem, starting with an arbitrary $\{Z_i\}_{i=1}^{i=n}$ we continue the process of increasing S and decreasing the number of positive Z_i's. The process stops when the vertices assinged $Z_i > 0$ form a complete graph.
Now if G is maximal, i.e. adding any more edges produces a K_p subgraph, then this complete subgraph must be K_{p-1} and the maximum value S can have on K_{p-1} is $\frac{\binom{p-1}{2}}{(p-1)^2}$. So

$$\frac{|E|}{n^2} \leq \frac{\binom{p-1}{2}}{(p-1)^2} \Rightarrow |E| \leq \frac{n^2(p-2)}{2(p-1)}$$

Exercise 5.1.8

1. Let $P = v_0v_1 \dots v_k$ be a maximal path in G then since $\delta(G) \geq 2, v_k$ has at least $\delta(G) - 1$ neighbours other than v_{k-1} and each of these must occur in P, else we could extend P to a longer path. Then the smallest cycle possible using the neighbours of v_k will be when all the neighbours of v_k appear before v_k in a sequence and in this case the length of the cycle formed is $\delta(G) + 1$. So the statement is true.

2. Since G doesn't contain three cycles, we have

$$\frac{n^2}{4} \geq e \geq \frac{nk}{2} \Rightarrow n \geq 2k$$

$2k$ is indeed the lowest bound as $K_{k,k}$ has four cycles and minimum degree k and $2k$ vertices.

3. Since G doesn't contain 3 cycles or 4 cycles, we have

$$\frac{1}{2}n\sqrt{(n-1)} \geq e \geq \frac{nk}{2} \Rightarrow n \geq k^2 + 1$$

4. Let G be a k-regular bipartite graph on n vertices with $k \geq 2$. Suppose $e = (u, v)$ is a cut edge, then $G - e$ has two components G_1, G_2 both bipartite with $u \in G_1$ and $v \in G_2$. Now say $G_1 = H_1 \sqcup H_2$ with $u \in H_1$.
Now every vertex in H_2 has degree k so G_1 has 0 mod (k) edges but every vertex in H_1 other than u also has degree k and u has degree $k-1$ which means that G_1 has -1 mod (k) edges. We obtain this contradiction by counting the number of edges in G_1 in two ways. The same can be done for G_2. Thus $G - e$ can't be disconnected. Therefore, G doesn't have cut edges.

5. Start with any vertex v and its three neighbours v_1, v_2, and v_3. Each of these neighbours have two other neighbours, so we can count at most six more vertices. If there is a vertex u that is not connected to any $v_i, i = 1, 2, 3$ then the path from v to u has length ≥ 3. So there are no other vertices $\Rightarrow n \leq 1 + 3 + 6 = 10$. The Petersen graph satisfies these conditions and as seen in Exercise 3.2.12 there is only one such graph on 10 vertices.

6. Pick any two vertices x, y. If there is no edge between them, then for every $u \in N(x)$ $\exists$ $v_u \in N(u) \cap N(y)$ and v_u is unique in the sense that for $w \in (N(x) \setminus \{u\})$ $v_w \neq v_u$ as u, w already have a common neighbour x. Thus, $d(x) \leq d(y)$. By symmetry $d(x) = d(y)$.
If x, y have an edge between them, then since there is no vertex of degree $(n-1)$ $\exists$ z such that y, z are not adjacent. For every $u \in N(x)$ $\exists$ $v_u \in N(x) \cap N(z)$ and $\exists$ $w_u \in N(v_u) \cap N(y)$ and as before this w_u is unique $\Rightarrow d(x) \leq d(y)$. By symmetry $d(x) = d(y)$.
Alternatively, since $\text{diam}(G) = 2$, we start with a vertex u of maximum degree, call it the generation 1 vertex. Its neighbours are generation 2 vertices and so on. There can be at most 3 generations. If there is a vertex in generation 3, then by the given condition this forces every generation 2 vertex to have a neighbour in generation 3 and so, every generation 3 vertex has maximum degree. Again backtracking, every generation 2 vertex now must have maximum degree. Also observe that the degree of every vertex is even as the neighbours pair up in order to satisfy the condition imposed.

7. Given an edge $e = (x, y)$, the vertices x, y have no common neighbour. So, $d(x) + d(y) \leq n$.Summing over all edges,

$$\sum_{(x,y)\in E} d(x) + d(y) \leq en \Rightarrow \sum_{x\in G} d(x)^2 \leq en$$

Using Cauchy-Schwarz inequality

$$\frac{4e^2}{n} \leq \sum_{x \in G} d(x)^2 \leq en$$

In this case, we have an equality at both ends, so for every edge $e = (x, y), d(x) + d(y) = n$. Now take any edge $e = (x, y)$ then $N(x)$ and $N(y)$ form a partition of G, so G is bipartite, say $G = V_1 \sqcup V_2$.
If V_1, V_2 have size n_1 and n_2 respectively then $n_1 + n_2 = n$ and $n_1 n_2 \geq e$ the latter inequality coming from the fact that G must be contained in the complete bipartite graph. By AM-GM inequality, we have

$$\frac{n_1 + n_2}{2} \geq \sqrt{n_1 n_2} \Rightarrow n_1 n_2 \leq \frac{n^2}{4}$$

So, $n_1 n_2 = \lfloor \frac{n^2}{4} \rfloor$.
If $n = 2k$, then $\frac{n^2}{4} = k^2$ and G must be $K_{k,k}$.
If $n = 2k + 1$, then $\frac{n^2}{4} = k(k + 1)$ so, $n_1 = k$ and $n_2 = k + 1$ and $G = K_{k,k+1}$.

8. Let $e = (x, y)$ be an edge, then the number of triangles t_e with e as an edge is $|N(x) \cap N(y)|$. We know

$$|N(x)| + |N(y)| - |N(x) \cap N(y)| \leq n \Rightarrow t_e \geq d(x) + d(y) - n$$

If we now sum this over all edges, the right hand side evaluates to $3t$ where t is the number of triangles as each triangle is counted thrice. So,

$$3t \geq \sum_{(x,y) \in E} (d(x) + d(y) - n) = \sum_{x \in G} d(x)^2 - ne \geq \frac{4e^2}{n} - ne$$

$$\Rightarrow t \geq \frac{e}{3n}(4e - n^2)$$

9. If $\delta(G) > \epsilon(G)$ then take $H = G$. Else, $\delta(G) \leq \epsilon(G)$ in which case, let v be a vertex with $d(v) = \delta(G)$ and look at the graph $G' = G - v$. It has $e - \delta(G)$ edges and $n - 1$ vertices where e is the number of edges in G and n is the number of vertices in G.

$$\frac{e - \delta(G)}{n - 1} - \frac{e}{n} = \frac{e - n\delta(G)}{n(n - 1)} \geq 0$$

as $\delta(G) \leq \epsilon(G)$. So $\epsilon(G') \geq \epsilon(G)$. The proof is completed by either induction (the case on 2 and 3 vertices is clear) or via a repetition of this procedure.

Chapter 6

Exercise 6.1.3

If G is a k-regular bipartite graph, then it must have the same number of vertices in both partitions (count the number of edges in two ways). Let $G = V_1 \sqcup V_2$, then for any $S \subseteq V_1$, since G is k-regular, $|N(S)| \geq |S|$. If not, then look at the subgraph $< S, N(S) >$, counting the edges through S we have $k|S|$ many edges, but counting the number of edges through $|N(S)|$ we would have at most $k|N(S)| < k|S|$ which is a contradiction. So, by Hall's theorem, there is a complete matching on V_1. Since G has the same number of vertices in its partitions, this mathching is a perfect matching of G.

Exercise 6.1.5

$\Rightarrow$ If such a subgraph H exists, then each x_i has d_i neighbours and two different x_i's don't have a common neighbour, so $\forall\ S \subseteq V_1, |N(S)| \geq \sum_{x\in S} d(x)$.
$\Leftarrow$ Suppose $\forall\ S \subseteq V_1 |N(S)| \geq \sum_{x\in S} d(x)$. Then, given two vertices $u, v \in V_1$, $N(u) \cap N(v) = \phi$ for $|N(u) \cup N(v)| = d(u) + d(v) - |N(u) \cap N(v)| \leq d(u) + d(v)$. So, two different vertices in V_1 have no common neighbour.
If $m = 1$, then the claim is clearly true.
Assume that the claim holds for $m - 1$ vertices in V_1. Then look at $G' = G - x_1$. $G' = V_1 \setminus \{x_1\} \sqcup V_2$ is a bipartite graph and $\forall\ S \subseteq V_1 \setminus \{x_1\}, |N(S)| \geq \sum_{x\in S} d(x)$ (as this doesn't change upon removing x_1). So, there is a subgraph H in G' such that $d_H(x_i) = d_i$ for $i = 2, 3 \ldots m$ and $d_H(y_i) \leq 1$ for $i = 1, 2, \ldots, n$ as G' satisfies the induction hypothesis.
Observe that if $y \in N(x_1)$ then $d_H(y) = 0$ as different x_i's have disjoint sets of neighbours. So, upon adding x_1 and its edges, the resulting graph H' is a subgraph of G with the stated properties.

Exercise 6.1.8

If G is bipartite, then any subgraph H is bipartite and so, inevitable one partition (which is an independent set) of H will contain at least half the vertices of H.
Conversly, suppose for every subgraph H of G there is an independent set containing at least half the vertices of H. Suppose G has an odd cycle $C = v_1v_2 \ldots v_{2k+1}$ of length $2k+1$. Then, we have an independent set C_1 with at least $k+1$ vertices. Each vertex in C_1 has degree 2 in C and two vertices in C can have at most one common neighbour since C is not a four cycle nor does it contain a four cycle. Also, no two vertices in C_1 are adjacent. So we can count at least one neighbour for every vertex in C_1 and this adds at least $k + 1$ more vertices to C contradicting the fact that C has only $2k + 1$ vertices. So, G doesn't contain odd cycles and hence is bipartite.

Exercise 6.1.12

Let G be a connected graph with no path or cycle of length more than 2. Since G is acyclic, it is a tree. Suppose there is a vertex v of degree $d \geq 2$ then every neighbour of v has degree 1, else we can create a path of length 3 or more. Since G is a tree, every edge must then be incident on v as if there is a vertex not adjacent to v then it must have a path to one of the neighbours of v which is not possible. So, G is a star graph.
Else, every vertex in G has degree 1 in which case G is K_2 which is also a star graph.
So, each component of such a graph is a star graph.

Exercise 6.1.16

Let M, M' be two matching and P a component in $M\Delta M'$. All edges in P are either from M or from M' but not from both. A vertex in P can't have degree ≥ 3 as then three or more edges meet at a point and inevitably two of them have to be from M or M'. So every vertex in P has degree ≤ 2. Also there are no isolated vertices in $M\Delta M'$ as every vertex in $M\Delta M'$ has an edge in one of M, M'.
Now, if every vertex in P has degree 2 then P is a cycle with edges alternating between M and M' (the edges must alternate as M, M' are matchings).
Else, P has vertices of degree 1 and in this case it must be a path with edges alternating between M and M'. (The fact that they must be cycles or paths is easily proven using induction on the number of vertices).

Exercise 6.2.7

1.$\Rightarrow$ If T has a 1-factor, then it has even number of vertices and $o(T - v) \leq 1\ \forall\ v \in T$. Since T has even number of vertices, it is necessary that $o(T - v) = 1\ \forall\ v \in T$.
$\Leftarrow$ T must have even number of vertices. Now, let u be a leaf and v its neighbour. In $T - v$, since u is isolated, every other component is even. Look at $T' = T \setminus \{u, v\}$. Each component of T' is an even tree and had exactly one edge to v. Let w be a vertex in a component C of T'. In $T - w$, there is an odd component M and let K be the component containing v. If M doesn't contain v, then M is contained in C for v is connected to every other component of T'. Upon removing u, v and other components of T', M stays odd and K remains connected with even number of vertices since there is only one edge from K to v as T is a tree and we removed an even number of vertices. Thus, $o(C - w) = 1$.
If M contains v then upon removing u, v and other components of T', M remains connected and odd. Thus each

component of T' satisfies the given condition. Now, by induction, each component of T' has a 1-factor. These together with edge (u,v) forms a 1-factor of T. Observe that when T has 2 vertices, then it does have a 1-factor.

2. For a given $S \subset V$, the best possible matching can saturate at most $|S|$ many odd components of $G-S$ using 1 vertex of S for each odd component in $G-S$ while giving perfect matchings in the even components of $G-S$ and matching all but one vertex in the remaining odd components of $G-S$, so $2\alpha'(G) \leq |V(G)| - (o(G-S) - |S|) = |V(G)| - d(S)$. Since this is true for any subset S we have $2\alpha'(G) \leq n - d$ where $d = \max_{S \subseteq V} d(S)$.
By the defective version of Tutte's 1-factor theorem, we have a matching H with $|V(H)| \geq |V(G)| - d$, so $2\alpha'(G) \geq |V(H)| \geq |V(G)| - d$. Thus, the result is true.

3. Observe that 1 regular graphs are matchings, so take $k \geq 2$. If $k = 2m$ then take $G = K_{2m+1}$. G has odd number of vertices, so no perfect matching.
Let $k = 2m+1$.Start with a vertex z and its k neighbours $\{v_1, v_2, \ldots, v_k\}$. For each $1 \leq i \leq k$, take $k-1$ copies of K_{2m+1}, say $G_1, G_2, \ldots G_{k-1}$. Label each vertex in $G_j, 1 \leq j \leq k-1$ from 0 to $k-1$. Now since $k-1$ is even, we can pair the G_j's and add an edge between vertices of positve label in each pair. An edge is added between the vertex labelled 0 in each G_j and v_i. We do this for all i and the resulting graph is k regular by construction.
There are $(k-1)k$ vertices connected to each v_i other than z and the only path between different v_i's is via z. So upon removing z we get k odd components and $k > 1$. Therefore, the graph we have constructed is k regular without a perfect matching.

4.For any $S \subset V$, the number of edges N from S to the odd components of $G-S$ is at most $k|S|$, so $N \leq k|S|$.
Now, since removing any $k-2$ edges doesn't disconnect the graph, we need at least $k-1$ edges from any odd components of $G-S$ to S. Now, for an odd component C of $G-S$ with m vertices and t edges to S, $\sum_{v \in C} d_G(v) = 2|E(C)| + t = km$. Since m is odd, $t \equiv k \mod (2)$, so $t \geq k$ for every odd component of $G-S$. Hence, $N \geq k o(G-S)$. Thus, by Tutte's 1-factor theorem, G has a perfect matching.

5. Follows from Exercise 6.2.7 4.

6. Q_n is an n regular graph on 2^n vertices. Let the vertices of Q_n be represented by n-tuples. We have a perfect matching on Q_n if we take the edges between vertices that differ only in the last coordinate. This is a matching as two different vertices can't have the same n tuple when we flip the last coordinate and every vertex has one such neighbour obtained by flipping the last coordinate. So, $\alpha'(Q_n) = 2^{n-1}$. Now, by the theorems of Gallai, Konig and Egervary, we have $\alpha(Q_n) = \alpha'(Q_n) = \beta(Q_n) = \beta'(Q_n) = 2^{n-1}$.

7. By Exercise 6.2.7 5, we have a matching M of G. In the edge deleted subgraph $G-M$ every vertex has degree 2, so every component is a cycle. Fix an orientation for each cycle (clockwise or counterclockwise). Since G is 3 regular it has an even number, say $2k$ vertices and hence $3k$ edges. If such a decomposition is possible we would get k P_4s. Now for each edge $e = (u,v) \in M$ there are cycles in $G-M$ C_1, C_2 such that $u \in C_1$ and $v \in C_2$. C_1 and C_2 may be the same and in this case u, v are not adjacent in $C_1 = C_2$. Let x be the vertex after u in C_1 and y the vertex after v in C_2. Then $P_e = xuvy$ is a path of length 4. We do this for every edge in the matching. If the collection $\{P_e : e \in M\}$ is a collection of edge disjoint paths, then we are done.
Take two edges $e, f \in M, e \neq f$. Suppose P_e and P_f have a common edge (a,b). Then vertices a, b are in the same cycle C of $G-M$ and take $e = (a,x)$ and $f = (b,y)$. While creating P_e we get b as the vertex after a in C. But while creating P_f we get a as the vertex after b but his would imply that C is a 2 cycle contradicting the simplicity of G. Therefore, $\{P_e : e \in M\}$ is a collectiong of k edge disjoint P_4's implying that G has the decomposition stated.

8. Let G have n vertices. Suppose such a decomposition is possible with say m r-factors. Then G has $\frac{nk}{2}$ edges which counted another way is $m.\frac{nr}{2}$. Therefore, $r|k$.
Conversely, suppose $k = mr$. We will induct on m. If $m = 1$, then it is trivial. Suppose, it is true for $k = mr - r; m > 1$, then observe that G has an r-factor by Hall's theorem (we can successively take r 1-factors), so once we remove one r-factor, we get a $k-r$ regular bipartite subgraph which, by induction hypothesis, decomposes into r-factors. These r-factors together with the one removed forms a decomposition of G into r-factors. Thus, by induction, G decomposes into r-factors.

9. Let T be a tree. If it doesn't have a matching we are fine. Suppose it has a perfect matching. Let u be a leaf and v its neighbour. Then every perfect matching must have the edge (u,v). Now, in $T-v$ there is only one odd component u and every other component is even. So, the matching restricted to these components is a perfect matching. Now each component has a smaller number of vertices than T. So, if each even component has a unique perfect matching, then M is uniquely determined. Observe that K_2 has a unique perfect matching. So, the proof follows by induction on the number of vertices of T.

10. Since trees don't have cycles, they don't have odd cycles. So, all trees are bipartite graphs. Further, they don't have isolated vertices. So $\alpha'(T) = n - \alpha(T) = n - k$.

11. (1.) If $\alpha(G) = 1$, then given any two vertices there is an edge between them. So, G is complete. Conversely, if G is complete then clearly $\alpha(G) = 1$.
(2.) If $\alpha'(G) = 1$, then any two edges have a common vertex. So, fix an edge $e = (u, v)$ then every other edge is incident on u or v. If e_1 is incident on u and e_2 is incident on v then they must have some other vertex w in common. Now if there is any other edge incident on either u or v, it will be disjoint from one of e_1 and e_2. Therefore, there are no other edges in G and G is K_3. Else, all edges are incident on exactly one of u, v and in this case G is a star graph. Conversely, if G is either K_3 or a star graph then clearly $\alpha'(G) = 1$.
(3.) If $\beta(G) = 1$ then every edge is incident on a single vertex v and G must be a star graph. Conversely, if G is a star graph then $\beta(G) = 1$.
(4.) If $\beta'(G) = 1$ then every vertex lies on a single edge e. Since an edge has exactly 2 vertices, G must be K_2. Conversely, if G is K_2 then clearly $\beta'(G) = 1$.

Chapter 7

Exercise 7.1.8

Let G be a directed multigraph (with loops and parallel edges) with a single source s, sink t, c a capcity function on the edges of G and f a feasible flow with value $v(f)$. We shall say that f is reduced if the flow on each edge is not more than the value of the flow.

Claim : Every feasible flow on G can be reduced, i.e. for every flow f with value $v(f)$, there is a reduced flow f', call it the reduction of f, with the same value.

Proof : Induction on the number of vertices n.
For $n = 2$, we would have 2 vertices s, t, forward edges $e_1, e_2, \dots e_m$ and backward edges $b_1, b_2, \dots b_k$. We may also have loops at both s, t. However, since loops don't contribute to the value of the flow, we may take the flow value on the loops to be 0. We construct a new flow f' as follows. If $v(f) = 0$, then the zero flow is the required flow. So assume $v(f) > 0$. If there is some b_j such that $f(b_j) > 0$, then there must be an edge e_i such that $f(e_i) > 0$ as $v(f) > 0$. Let $\epsilon = \min\{f(e_i), f(b_j)\}$. Consider the new flow f_1 given by

$$f_1(e) = \begin{cases} f(e) & \text{if } e \neq e_i, b_j \text{ and is not a loop,} \\ f(e) - \epsilon & \text{if } e = e_i \text{ or } e = b_j, \\ 0 & \text{if } e \text{ is a loop.} \end{cases}$$

Observe that f_1 has the same value as f. Now, one of $f_1(e_i), f_1(b_j)$ is 0. We may continue this way. The process stops when each b_j has 0 flow. This process terminates as at each step one or more edges get 0 flow but the flow value stays positive. Let the resulting flow be f'. By construction f' has the same value as f. Further, all the backward edges and loops have 0 flow. It follows from $v(f') = \sum_{i=1}^{i=m} f'(e_i) - \sum_{j=1}^{j=k} f'(b_j) = \sum_{i=1}^{i=m} f'(e_i)$ that $f'(e) \leq v(f')$ for each edge e.
Thus f' is the required flow and the claim is true for $n = 2$.

Assume that the claim is true for every multigraph on n' vertices for $n' < n$.
G is a graph on n vertices. Collapse all vertices in $N(t) \setminus \{t\}$ to a single vertex w, retaing all the loops and parallel edges created and call this new graph G'. It is possible that $G' = G$. We have the map

$$\begin{aligned} \phi \colon E(G) &\to E(G') \\ (a, b) &\mapsto \overline{(a, b)} \end{aligned}$$

where $\overline{(a, b)}$ is the image of edge (a, b) after collapsing all of $N(t) \setminus \{t\}$ to w. Since we have retained all loops and parallel edges, ϕ is a bijection. Using ϕ we may define the capacity function c' on G' by $c'(e) = c(\phi^{-1}(e))$.
Now, if S is the the set of all feasible flows on G and S' the set of all feasible flows on G', then ϕ induces the map

$$\begin{aligned} \psi \colon S &\to S' \\ f &\mapsto g \colon e \mapsto f(\phi^{-1}(e)) \end{aligned}$$

ψ is well defined as the capacity on $\phi(e)$ is the same as that on $e \ \forall\ e \in E(G)$, by the construction of c' on G'.
Since ϕ is a bijection, it follows that ψ is also a bijection.
In G, an edge e is a loop at t if and only if $\phi(e)$ is a loop at t in G' and e leaves (enters) t if and only if $\phi(e)$ leaves (enter) t in G'. Thus, by definition of $\psi(f)$, we also have $v(\psi(f)) = v(f)$, i.e. ψ preserves the value (note that the value of a flow is also given by $v(f) = \sum_{e \in I(t)} f(e) - \sum_{e \in O(t)} f(e)$ where $I(t)$ is the set of edges that enter t and $O(t)$ those that leave t).

Finally, a flow f in G' can be broken down into two flows : one from w to t, and another from s to w, and in case s, t are adjacent, the latter case doesn't occur. Let $F_1 = G' - t$ and F_2 be the subgraph induced by w, t. Define the respective capacities to be the restriction of that on G', and let $f_1 = f|_{F_1}, f_2 = f|_{F_2}$. Since w is an intermediate vertex, $v(f_1) = v(f_2)$.
In case s is the only neighbour of t, then $G' = G$ and the flow from s to s is redundant. In this case, we may ignore all edges that are not between s, t and effectively go back to the $n = 2$ case as all other edges don't contribute to the flow. So, we may assume that s is not the only neighbour of t.
Both F_1, F_2 have smaller number of vertices, so induction hypothesis applies (in the case where s, t are adjacent, we can ignore F_1). So, a reduction of f_1, say g_1 and a reduction of f_2, say g_2 exist. Combining the two gives a flow g on G' with the same value as f_1. Further, since $v(f) = v(f_1) = v(f_2)$, it follows that g is a reduction of f. Now, using ψ, we can look

at the preimage of g to get a reduction of f.
Thus, by induction the claim is true. ■

Now, let G be a graph with capacity function c. Suppose the min-cut is finite, then the sup-flow S is also finite. Further, given a flow f, let f' be its reduction. Since $v(f) \leq$ min-cut, for each edge e, we have $f'(e) \leq v(f) \leq$ min-cut.
Given $n \in \mathbb{N}, \exists f_n$ flow with $S - \frac{1}{n} \leq v(f_n) \leq S$, then using the reduced flow f'_n, we have $S - \frac{1}{n} \leq v(f'_n) \leq S$ and the sequence $\{f'_n(e)\}_{n\geq 1}$ is bounded for each edge e. Since there are a finite number of edges, we may get hold of a subsequence which converges for each sequence. Let the limit be $f(e)$. f is then a flow on G because limits commute with sums and differences. Further, $v(f) = S$.
Thus, if the min-cut is finite, the sup-flow is attained.

Exercise 7.1.12

Given an $s-t$ path P in the underlying undirected graph, define $c_f(P) = \min\{i(e) : e \in P\}$ where $i(e) = c(e) - f(e)$ if e is a forward edge and $i(e) = f(e)$ if e is a backward edge. Note that f takes nonnegative values since the graph is directed.
Now, define a new flow f' by

$$f'(e) = \begin{cases} f(e) & \text{if } e \notin P, \\ f(e) + c_f(P) & \text{if } e \in P \text{ is a forward edge,} \\ f(e) - c_f(P) & \text{if } e \in P \text{ is a backward edge.} \end{cases}$$

Observe that f' is a flow on the given graph and $v(f') = v(f) + c_f(P)$.
So, given a flow, it would be maximal if there is no such residual path. Now define $S = \{v : \exists\ s-t \text{ path which has positive residual capacity }\}$. Now if f is maximal then $t \notin S \Rightarrow [S, S^c]$ is a cut. Now if e is an edge from S to S^c then $f(e) = c(e)$ by construction of S and if e is directed from S^c to S then $f(e) = 0$.
$v(f) = f(S, S^c) - f(S^c, S) = c(S, S^c)$. Therefore max-flow = min-cut.
Observe that $\forall\ x \in S,\ f(x, V) - f(V, x) = 0\mathbf{1}[x \neq s] + v(f)\mathbf{1}[x = s]$ thus, $v(f) = f(S, V) - f(V, S) = f(S, S) + f(S, S^c) - f(S, S) - f(S^c, S)$

Exercise 7.1.13

Orient each edge of G both ways and give the same capacity to both as the original edge's capacity to create a directed graph H. We have a maximal flow f in H whose value is the min-cut of H which by our construction of H is the same as the min-cut of G since while calculating the value of any cut in H we require only certain forward edges.
More elaborately, if $[S, S^c]$ is a cut in G then $[S, S^c]$ is a cut in H and their cut value is the same since we are considering forward edges. Conversely, if $[S, S^c]$ is a cut in H then it is a cut in G and since H involves both orientations for all edges of G it follows that its cut value is the same in G.
Now define f' on G by $f'(e) = f(e) - f(-e)\ \forall\ e \in G$. f' clearly satisfies the capacity constraints and antisymmetry since f takes nonnegative values. Further, $v(f') = \sum_{e:e^-=s} f'(e) = \sum_{e:e^-=s} f(e) - \sum_{e:e^+=s} f(e) = v(f)$. Therfore max-flow min-cut theorem holds for undirected graphs.

Exercise 7.1.14

See Exercise 7.1.12.

Exercise 7.1.15

Yes. Multiply each capacity to clear the denominators to get integer valued capacitites c'. Now given a maximal flow f for c we can multiply by the same factor to get a flow for c' and this is maximal as we can divide and get back to c. Since the Ford-Fulkerson algorithm terminates and gives an integral flow, it terminates when the capacities are rational and gives a rational valued flow.

Exercise 7.2.1

- For the complete graph K_n, we need to remove $n-1$ vertices or edges to disconnect the graph or reach K_1. We have $\kappa(K_n) = \lambda(K_n) = n - 1$.
- For the path graph P_n on n vertices it is clear that $\kappa(K_n) = \lambda(K_n) = 1$.
- For the cycle graph we need to remove at least two vertices or edges to disconnect the graph, so $\kappa(K_n) = \lambda(K_n) = 2$.
- For the Cayley graph edge and vertex connectivity depends on the group.
- For the Petersen graph G, since it is three regular $\lambda(G) \leq 3$. However, removing any two edges doesn't disconnect the graph. Thus, $\lambda(G) = 3$. Now, $\kappa(G) \leq \lambda(G) = 3$ but removing any 2 vertices doesn't disconnect G, hence $\kappa(G) = \lambda(G) = 3$.

Exercise 7.2.4

Let $[S, S^c]$ be a min-cut, i.e. min-cut$(s,t) = |E \cap (S \times S^c)|$ since all capacities are 1. So, $C(s,t) \leq$ min-cut(s,t). Conversely, if E' is a minimum set of edges that disconnect s and t, the take S to be the component of s in $G - E'$ and T to be the component containing t. Clearly $t \notin S$. Now if K is a component different from S, T, then we can add at least one edge from K to a different component as G is connected and this keeps s and t disconnected but gives a smaller cut. Hence, there are exactly 2 components in $G - E$ and $T = S^c$. Thus, S, S^c are the two components of $G - E'$. So, the only edges from S to S^c are those in $E' \Rightarrow$ min-cut$(s,t) \leq C(s,t)$. Hence, min-cut$(s,t) = C(s,t)$.

Exercise 7.2.5

Let $S = \{v | \exists$ an $s - v$ directed path P such that$f(e) > 0 \; \forall \; e \in P\} \cup \{s\}$. $S \neq \phi$ since $s \in S$. If $t \notin S$, then $[S, S^c]$ is an $s - t$ cut. Every edge going out of S has 0 flow value by definition $\Rightarrow v(f) = f(S, S^c) - f(S^c, S) = 0 - f(S^c, S) \leq 0$ a contradiction to $v(f) > 0$. Therefore, $t \in S$ and hence such a path exists in G_f.

Exercise 7.2.7

Let $\lambda(s,t) =$ maximum number of edge disjoint paths from s to t then, min-cut$(s,t) \geq \lambda(s,t)$ for every edge cut must contain at least one edge from each edge disjoint $(s - t)$ path.
Assign a capacity of 1 on each edge of the graph G with source s and sink t. Now max-flow$(s,t) \geq \lambda(s,t)$ by Max-flow Min-cut theorem (or directly since we can send a flow of 1 along each path in a maximum set of edge disjoint $s - t$ paths). Note that min-cut in this graph corresponds to the minimum number of edges to be removed to disconnect s and t. Now the flow value for max-flow are in $\{0, 1\}$. The result in Exercise 7.2.5 applies to undirected graphs as well (the proof is similar), so take a path P where the flow has a constant 1. Now remove the edges of P to form the graph $G - P$ which has flow value strictly less than the previous flow value. Continuing this way we get max-flow(s,t) many paths and they are all edge disjoint. So, max-flow $(s,t) \leq \lambda(s,t)$. By Max-flow Min-cut theorem, the result holds.

Exercise 7.2.8

NOTE : We have to assume that there is no edge (s,t) in G for otherwise vertex connectivity is not defined.
Let $\kappa(s,t) =$ maximum number of vertex disjoint paths from s to t and $C(s,t)$ the minimum number of vertices to be removed to disconnect s and t. Then $C(s,t) \geq \kappa(s,t)$ since we need to remove at least one vertex from each path in the maximal set of vertex disjoint $s - t$ paths. Create a new graph G' defined as follows:
$V(G') = \{s,t\} \cup \{x_1, x_2 : x \in V(G) \setminus \{s,t\}\}$ and
$E(G') = \{(s, x_1) : (s,x) \in E(G)\} \cup \{(x_2, t) : (x,t) \in E(G)\} \cup \{(x_2, y_1) : (x,y) \in E(G)\} \cup \{(x_1, x_2) : x \in V(G)\}$.
G' is an undirected graph. Observe that there is a natural bijection between paths in G' and those in G and this bijection is such that two paths in G' are edge disjoint if and only if the corresponding paths in G are vertex disjoint. So, $\kappa(s,t) = \lambda_{G'}(s,t)$.
Now given a set of edges F in G' that disconnect s and t, for each edge $e \in F$, there is a vertex $x_e \in V(G) \setminus \{s,t\}$ such that e contains one of x_1, x_2. Then removing all such $\{x_e : e \in F\}$, we disconnect s and t in G for if there was an $s - t$ path in G we would get an $s - t$ path in G'. So, $\lambda_{G'}(s,t) \geq C(s,t)$.
Conversely, given a set of vertices H in G that disconnect s and t in G, look at the set of edges $F = \{(x_1, x_2) : x \in H\}$. Then as above F disconnects s and t in G'. So, $C(s,t) \geq \lambda_{G'}(s,t)$. Therefore, $C(s,t) = \lambda_{G'}(s,t) = \kappa(s,t)$ as required.

Exercise 7.4

1. Firstly, the answer is the same for both parts as given a path satisfying (a), it satisfies (b) and conversely given a path P in (b), we can reduce it to a path in (a) by considering the last vertex of P in S and the first vertex of P in T. Observe that this way a set of paths in (a) gives a set in (b) and vice versa. So, the maximum in both (a) and (b) is the same. Now collapse all vertices of S to a single vertex s and all vertices of T to a single vertex t. Now given two vertex disjoint paths as in (a), they correpond to vertex disjoint paths in this new graph H and given two vertex disjoint paths in H, their preimage in G are vertex disjoint. Observe that any $s - t$ path in H must come from a path in G going from S to T. A similar reasoning shows that the minimum number of vertices to be removed to disconnect S and T in G is the same as that to disconnect s and t in H. Now the result follows by Menger's theorem applied to H.

2.Suppose $\kappa(G) = 1$, then upon removing some vertex v, $G - v$ has at least 2 components and since $\Delta(G) \leq 3, G - v$ has at most 3 components. Now if $G - v$ has 2 components C_1, C_2 then there is a component, say C_1, such that v has only on edge e going to C_1. Then $G - e$ is disconnected, so $\lambda(G) = 1$. Else, $G - v$ has 3 components C_1, C_2, C_3 with exactly one edge e_i of v going to each $C_i, i = 1, 2, 3$.
If $G - e_1$ is connected then there are edges between C_1 and at least one of C_2, C_3 but not both as otherwise $G - v$ would be connected. Suppose there are edges between C_1 and C_2. Then there are no edges between C_3 and C_1 or $C_2 \Rightarrow G - e_3$ is disconnected. Else $G - e_1$ is disconnected. Either way, $\lambda(G) = 1$.
Suppose $\kappa(G) = 2$, say removing vertices v, w disconnects the graph and there is no cut vertex. Then $G - v$ is connected and $G - \{v, w\}$ is disconnected. As reasoned above, $(G - v) - w$ has at most 3 components. Since $G - w$ is connected, v must have edges to all these components. First, suppose $G - \{v, w\}$ has two components C_1, C_2. Then both v, w have

edges to C_1, C_2. Since $\Delta(G) \leq 3$, there is a single edge e_v from v to a component C_v and a single edge $e - w$ from w to a component C_w. Clearly, $G - \{e_v, e_w\}$ is disconnected and hence $\lambda(G) = 2$.
Else, $G - \{v, w\}$ has 3 components, C_1, C_2, C_3. Since $\kappa(G) = 2$, each of v, w must have one edge to each $C_i, i = 1, 2, 3$ as if say v has no edge to C_3, then w must have edges to C_3, however removing w disconnects the graph. Now pick a component, say C_1 and remove the edges e_1, e_2 from v, w to C_1. There is only one such edge since $\Delta(G) \leq 3$. Then, $G - \{e_1, e_2\}$ is clearly disconnected. So, $\lambda(G) = 2$.
Finally, if $\kappa(G) = 3$, then it is clear that $\lambda(G) = 3$ by Whitney's theorem.

3. Fix an edge $e = (s, t)$. Then, there are at least two cycles C_1, C_2 such that $C_1 \cap C_2 = \{e\}$.So, we have at least 3 edge disjoint $s - t$ paths. By assumption, every edge of G is part of some cycle, so there are no cut edge. If e, f are two edges such that $G \setminus \{e, f\}$ is disconnected, then there must be exactly 2 components in $G \setminus \{e, f\}$ for if there are three or more components, then there will be two components that have an edge between them and this edge must be one of e, f. Adding this reduces the number of components by 1, but keeps G disconnected which means that the other edge is a cut edge, which is a contradiction.
So, $G \setminus \{e, f\}$ has exactly two components G_1, G_2 and e, f are the only edges between them. Now, any cycle C containing e must have another edge going between the components, but f being the only other edge between G_1, G_2, C must contain f. This contradicts the assumption that there are at least two cycles whose intersection is e.
Thus G is 3-edge-connected, i.e. removing 1 or 2 edges doesn't disconnect G, but removing 3 edges may or may not disconnect G (in the latter case, G is 4-edge-connected).

4. See Exercise 7.2.1.

5. For $F \neq \emptyset$, we need to show $F = [S, S^c]$ for some $S \subset V(G)$ if and only if F contains an even number of edges from every cycle in G.
Suppose $F = [S, S^c]$ for some S and let C be a cycle in G. If $C \subseteq S$ or $C \subseteq S^c$, then F contains no edges from C. Now collapse all of S to a vertex α and all of S^c to a vertex β. The resulting simple graph is K_2. In this graph C reduces to C' which repeatedly traverses the edge(α, β). If it helps, you may assume a loop at both α, β, then C' is such that every time an edge of C goes from S to S^c, C' goes from α to β and vice versa. Every time an edge of C stays in S or S^c, C' uses the loop at α, β respectively. Since C is a cycle, C' must eventually come back to α, assuming it starts at α. In doing so, it has used the edge (α, β) an even number of times. This is exactly the number of times an edge from F appears in C. Thus C has an even number of edges from F.
Conversely, suppose F contains even number of edges from each cycle. Let $e = (u, v) \in F$. If P is any $u - v$ path in G disjoint from e, then $P + e$ is a cycle, so F contains at least one other edge from $P + e$ since $e \in F$. Thus, F contains an edge from P. So, P is not a path in $G - F$. Since any $u - v$ path in $G - F$ is a path in G, we conclude that there is no $u - v$ path in $G - F \Rightarrow G - F$ is disconnected.
Now, let $S_1, \ldots, S_t$ be components of $G - F$. Collapse each S_i to a vertex $v_i, i = 1, 2, \ldots, t$. Let the resulting simple graph be G'.

<u>Claim :</u> G' is bipartite.

<u>Proof :</u> Suppose G' has an odd cycle $C = v_1 v_2 \ldots v_r v_1, r \geq 3$, by relabelling. Each edge $v_i v_{i+1}, 1 \leq i \leq r$, with the understanding that $r + 1 = 1$, corresponds to an edge $(a_i, b_i) \in F$ where $a_i \in S_i, b_i \in S_{i+1}$. This edge is in F since $S_1, \ldots S_t$ are the components of $G - F$. With this notation, observe that $b_i, a_{i+1} \in S_i$, so there is a simple path from b_i to a_{i+1} in S_i. Using these paths we get a cycle C' $a_1 \to b_1 \to \cdots \to a_2 \to b_2 \to \cdots \to a_3 \to \cdots \to b_r \to \cdots \to a_1$ in G, where all edges other than those of the form (a_i, b_i) are not in F.
Now if C is odd then the edges of C' in F are precisely those of the form (a_i, b_i) and are odd in number which is a contradiction.
Thus, G' has no odd cycle, and hence is bipartite. ■

Now, let $G' = X \sqcup Y$. X, Y correspond to two disjoint sets of components in $G - F$. Let the vertices in the pullback of X be S, then those in the pullback of Y are S^c and the edges of F are between S and S^c since the edges of G' are between X and Y. Thus, $F = [S, S^c]$ as required.

6.Adding a new vertex can only increase the connectivity. Now let S be a set of vertices which disconnects G' or $G' - S = K_1$. If $y \in S$, then $G - (S \cap G)$ is disconnected or K_1 and hence, $|S| \geq k$. If $y \notin S$, then $S \subset G$. If $G - S$ is disconnected or K_1, then $|S| \geq k$, else $G - S$ is connected and not K_1 and hence, the components of $G' - S$ are $\{y\}$ and G. Since y has at least k neighbours, $|S| \geq k$ as S must contain all neighbours of y.

7.The first implication is obvious.

If there is no cycle containing x, y then there is a unique path from x to y and removing any intermediate vertex disconnects the graph, else (x, y) is an edge and since it is part of no cycle, it is a cut edge. So, removing x or y disconnects the graph since the graph has at least 3 vertices, so G is not 2-vertex-connected. Conversely, if for every x, y there is a cycle containing them, then upon removing any vertex z, G stays connected. This is because, given $x, y \neq z$, if the cycle containing x, y doesn't contain z then it is a cycle in $G - z$, else, since there are at least two vertex disjoint paths from x to y, and removing z can't disconnect both of them. So, G is 2-vertex-connected. This clears the second implication.

If $\delta(G) \geq 1$ and every pair of edges lie in a common cycle then there are no isolated vertices. If G is disconnected, then for two edges from 2 different components there is no cycle containing both of them which is a contradiction, so G is connected. Now, suppose v is a cut vertex. Then $G - v$ has at least two components, take two of them, say C_1, C_2. There is an edge e_i from v to C_i for $i = 1, 2$. So, there is a cycle C that contains e_1, e_2 but this contradicts the assumption that C_1, C_2 are distinct components as C gives a path from vertices in C_1 to C_2. Thus, G doesn't have a cut vertex and hence is 2-vertex-connected.

Conversely, suppose that for every pair of vertices in G there is a cycle containing both of them. Let $e_1 = (u, v)$ and $e_2 = (x, y)$ be two different edges. If a vertex is common, say $u = x$, then using a $v - y$ path not containing x (it exists, by assumption), we can construct a cycle containing e_1, e_2.
So, assume $e1, e2$ are disjoint. By assumption, there are at least two vertex disjoint $u - x$ paths, take two of them, say P_1, P_2. Similarly take two vertex disjoint $v - y$ paths Q_1, Q_2.

- Suppose $v, y \in P_1$, then there is a path P in P_1 from v to y. Clearly, $e_1 + P + e_2 + P_2^-$ is then the required cycle where P_2^- is the reverse of P_2.
- Suppose neither v nor y are in P_1.
  - If one of Q_1, Q_2 is disjoint from P_1, then $e_1 + Q_1 + e_2^- + P_1^-$ is the required cycle.
  - Else both Q_1, Q_2 intersect P_1. In this case, start at u and go along P_1 until the first vertex of P_1 which is in Q_1 or Q_2, say w and it occurs in Q_1 (w may be u). From w go along Q_1 until the last vertex of $Q_1 \cap P_1$ before v (note that we are in some sense treating these paths as directed paths), say z. It is possible that $z = w$.
    * If there are no vertices of Q_2 in P_1 after z, then go along P_1 till v and then jump to y via edge e_2. Now from y traverse along Q_2 till x and then go back to u via edge e_1. This gives us a closed walk. Observe that while going from u to y, we haven't repeated any vertices. Further, while returning via Q_2, we don't cross any vertices between z, v by assumption, nor any vertices between w, z as Q_1, Q_2 are disjoint. Further, since w is the first vertex of P_1 in one of Q_1, Q_2, we don't cross any vertices between u, w either. Thus, what we have constructed is a cycle that contains e_1, e_2.
    * If there are vertices of Q_2 in P_1 between z, v, then from z continue along Q_1 till y. From y, hop to v via e_2. Then, starting from v go along P_1 till the intersection with Q_2 and then go along Q_2 from that vertex till x. Then go back to u via e_1. This is a closed walk. Further, one can see that, as above, it is actually a cycle containing e_1, e_2.
- Finally, we are left with the case where P_1 contains one of x, y, say x and P_2 contains the other y. Now, using P_1, we get hold of an $x - v$ path R_1 and using P_2 we get hold of a $u - y$ path R_2. Then $R_1 + e_2 + R_2 + e_1$ is the required cycle.

Thus we have dealt with all possible cases and in each one there is a cycle containing both e_1, e_2.

8. By Exercise 7.2.5, there are directed paths with constant positive flow whenever f has a positive strength. Let P be one such path and w the minumum flow value on the edges of $P; w > 0$. Consider the elementary flow of value w on P. Removing this flow f_1, the value of the original flow reduces by w. Keep repating this procedure to get elementary flows $f_1, \ldots f_k$. This process terminates since there are a finite number of directed $s - t$ paths and each path is considered at most once. At each stage, if the resulting flow value is not zero, we find a new elementary flow and reduce the flow value, but keep it nonnegative, else the flow value is zero and the process stops. When the process stops, the resulting flow g has zero value.
Since we reduced the flow value by the value of the elementary flows at each stage, it is clear the the value of the original flow is the sum of the values of the elementary flows obtained in this process. Also, the flow value at each edge is the sum of the corresponding values in each f_i and whatever is left comes from g. Thus $f = f_1 + f_2 + \cdots + f_k + g$.

9. From Exercise 7.4.8, $f = f_1 + f_2 + \cdots + f_r + g$ for some elementary flows $f_i, 1 \leq i \leq r$ and a flow of strength 0 g. Each f_i has an integral value k_i and corresponds to a path P_i in G. Consider the (multi) set of k paths $\{P_1, P_1, \ldots, P_1, P_2, P_2, \ldots, P_2, \ldots, P_r, P_r, \ldots, P_r\}$ where each P_i occurs k_i times. There are k many paths since $k_1 + \cdots + k_r = k$. Also, from the definition of elementary flow, for each edge that occurs in one of P_i, the given condition is satisfied. For edges not in any P_i, the given condition is satisfied anyway since f takes nonnegative values.

10. Construct a new graph G' which has vertex set $V' = \{s,t\} \cup \{x_1, x_2 : x \in V \setminus \{s,t\}\}$ and directed edge set $E' = \{(s,x_1) : (s,x) \in E\} \cup \{(x2,t) : (x,t) \in E\} \cup \{(y_2,x_1) : (y,x) \in E\} \cup \{(x_1,x_2) : x \in V \setminus \{s,t\}\}$. Assign capacity

$$c(e) = \begin{cases} c(x) & \text{if } e = (x_1,x_2), \\ c(s,x) & \text{if } e = (s,x_1), \\ c(x,t) & \text{if } e = (x_2,t), \\ c(x,y) & \text{if } e = (x_2,y_1). \end{cases}$$

Now given any cut in G we can take the corresponding edges to form a cut in G' both having the same cut value, and given any cut in G' we can get a cut in G of the same cut value. So min-cut(G) = min-cut(G'). The same thing holds for flows, given a flow f in G, create a flow

$$f'(e) = \begin{cases} f((x,y)) & \text{if } e = (x_2,y_1), \\ f((s,x)) & \text{if } e = (s,x_1), \\ f((x,t)) & \text{if } e = (x_2,t), \\ \displaystyle\sum_{e:e^+=x} f(e) & \text{if } e = (x_1,x_2). \end{cases}$$

f' is a flow on G' since it clearly satisfies the capcity constraints and conservation laws as f does. Further, $v(f') = v(f)$. Thus given a flow on G we can construct a flow on G' with the same flow value.
Conversely, given a flow f' on G' construct the flow

$$f(e) = \begin{cases} f'(s,x_1) & \text{if } e = (s,x), \\ f'(x_2,t) & \text{if } e = (x,t), \\ f'(x_2,y_1) & \text{if } e = (x,y). \end{cases}$$

f is a flow on G since capacity constraints on edges are readily satified and for vertices, since f' satisfies the corresponding edge capcity constraints, f satisfies vertex capacity constraints as well. Further, flow is conserved since f is a flow. Also, $v(f) = v(f')$ by construction. Thus given a flow f' on G' we can construct a flow f on G of the same value. Hence, max-flow(G) = max-flow(G').
Finally, since we have already proven the theorem for directed graphs with capacity on edges, it follows that the max-flow min-cut equality holds for G as well.

11.Hall's theorem by max-flow min-cut theorem :
Given a bipartite graph $G = G_1 \sqcup G_2$ that satisfies Hall's condition. Add two new vertices s,t with edges $\{(s,x) : x \in G_1\} \cup \{(y,t) : y \in G_2\}$ to form a new graph H. Now, let S be a minimum set of vertices that disconnect s and t, say $S = S_1 \sqcup S_2$ with $S_i \subseteq V(G_i), i = 1,2$. Suppose $S_2 \neq \emptyset$.
Now, if $y \in S_2$ is not a neighbour of any $x \in G_1$, then $S \setminus \{y\}$ is a smaller cut, so every $y \in S_2$ is a neighbour of some $x \in G_1$. Now, let $T = \{x \in G_1 : x \text{ has a neighbour in } S_2\}$. If $T \subseteq S_1$, then S_1 is a smaller cut. So, $T \nsubseteq S_1$. Let $T' = T \setminus S$. Then $\forall\, x \in T', N(x) \subseteq S_2$ for if $\exists\, x \in T', y \in G_2 \setminus S_2$ such that $x \sim y$, then $sxyt$ is an $s-t$ path in $G \setminus S$ a contradiction. Thus $N(T') \subseteq S_2$. By Hall's condition, $|N(T')| \geq |T'|$. Then, $|S| = |S_1| + |S_2| \geq |S_1| + |T'|$ and $S_1 \cup T'$ is a cut which contradicts minimality of S. Observe that $S_1 \cup T'$ is a cut since removing it from H removes S_1 and all neighbours of S_2 in G.
So, $S_2 = \emptyset \Rightarrow$ minimum vertex cut comes from G_1.
Now, if $S_1 \subset G_1$, then $\exists\, x \in G_1 \setminus S_1$ and its neighbour $y \in G_2$ which would create an $s-t$ path in H. So $S_1 = G_1$. Hence the minimal set of vertices which disconnect H is G_1, and $\kappa(H) = |G_1|$.
By Menger's theorem (a consequence of max-flow min-cut theorem), there are $|G_1|$ many vertex disjoint $s-t$ paths in H and since G is bipartite, neighbours of s are in G_1, these vertex disjoint paths reduce to a complete matching of G_1 proving Hall's theorem.
The defective version of Hall's theorem follows from Hall's theorem, so we can modify the above proof for the defective version.

Tutte's 1-factor theorem by Hall's theorem:(sketch of the following proof is indicated in exercise 3.3.13 of the book Introduction to Graph Theory by Douglas B. West.)
Let G be a graph satisfying Tutte's condition. If we remove one vertex of G, then there is necessarily one odd component. So, let T be a maximal set such that $o(G-T) = |T|$. Such a set exists as single vertices v satisfy $o(G-v) = 1$.
If there is an even component C in $G-T$, then let v be vertex of C, $C-v$ necessarily has an odd component as $C-v$ has odd number of vertices. So, $T \cup \{v\}$ has exactly $|T|+1$ odd components by Tutte's condition contradicting maximality of T. Thus, all components of $G-T$ are odd.
Next, let C be a component of $G-T$, and $x \in C$ a vertex. Look at $C-x$. Since $C-x$ has even number of vertices,

and since T is maximal, each component of $C - x$ is even. Now let C' is a component of $C - x$, and $S \subset C'$, then in $G - (T \cup S \cup \{x\})$ we have $o(C' - S) + |T| - 1$ odd components. By Tutte's condition and maximality of T, it follows that $o(C' - S) + |T| - 1 \leq |T| + |S| + 1 - 1 \Rightarrow o(C' - S) \leq |S| + 1$. Now, C' has even number of vertices, so it is not possible that $o(C' - S) = |S| + 1$ as then C' would have an odd number of vertices. So, $o(C' - S) \leq |S|$. Thus, each component of $C - x$ satisfies Tutte's condition, and hence, $C - x$ satisfies Tutte's condition.

Here x was arbitrary, so we have proved that for each component C of $G - T$, and any vertex $x \in C, C - x$ satisfies Tutte's condition.

Next construct the bipartite graph H by having the vertices of T on one partition X and a vertex v_C for each component of $G - T$ on the other partition Y. Draw an edge between $v \in T$ and v_C if $N(v) \cap C \neq \emptyset$.

Let $S \subseteq Y$, and suppose $|N(S)| < |S|$. Look at the corresponding components of G_T in G. Then removing $N(S) \subset T$ results in more than $|N(S)|$ odd components violating Tutte's condition (note that each component of $G - T$ is odd).

Thus it follows that $|N(S)| \geq |S|$, hence H satisfies Hall's condition. Thus, there is a matching that saturates Y. Since $|T| = |Y|$, this matching is a perfect matching.

Now, we can go back to G, and there is a set of independet edges that saturate T. For each $t \in T$, there is a component C_t of $G - T$ and a vertex $x_t \in C_t$ such that the edge (t, x_t) is included in this independent set.

As shown above, each $C_t - x_t$ satisfies Tutte's condition. So, we may complete the proof by induction on the number of vertices. The base case (on 2 or 4 vertices) is quite trivial.

Thus, we have proved Tutte's theorem from Hall's theorem. In the induction process, we get layers of bipartite graphs, so we may adjust the proof so as to use max-flow min-cut theorem.

Chapter 8

Exercise 8.1.2

If there exists a graph homomorphism $\phi\colon G \to K_k$, then assign colours $\{1, 2, \ldots, k\}$ to the vertices of K_k. For each vertex x of G assign the colour of $\phi(x)$. This gives a colouring of G since if $x \sim y$ in G, then $\phi(x) \sim \phi(y)$ and have different colours, hence x, y have different colours. Thus, G is k-colourable.

Exercise 8.3

1.

- Given k colours, each vertex of K_n must have a different colour. So, there are ${}^kP_n$ possibilities with the convention ${}^kP_n = 0$ if $k < n$. Then the chromatic polynomial is clearly $P(x) = \prod_{i=0}^{i=n-1} (x - i)$.
- Let C_n denote the cycle graph with n vertices and P_n the path graph with n vertices. Let $f_n(x), g_n(x)$ be their respective chromatic polynomials. By contraction-deletion principle, we have
  $f_n(x) = g_n(x) - f_{n-1}(x)$ by considering any edge and
  $g_n(x) = xg_{n-1}(x) - g_{n-1}(x)$ by considering a leaf edge in P_n.
  Since P_n is a tree, by Exercise 8.3 3., $g_n(x) = x(x-1)^{n-1}$. So,
  $f_n(x) = x(x-1)^{n-1} - f_{n-1}(x) = x(x-1)^{n-1} - x(x-1)^{n-2} + f_{n-2}(x) = x(x-1)^{n-2}(x-2) + f_{n-2}(x)$. Now, since $f_3(x) = x(x-1)(x-2)$, we have, for odd n

  $$f_n(x) = x(x-2)[(x-1) + (x-1)^3 + \cdots + (x-1)^{n-2}] = (x-1)^n - (x-1)$$

  For even n, $n \geq 4$, we have

  $$f_n(x) = x(x-1)^{n-1} - f_{n-1}(x) = x(x-1)^{n-1} - (x-1)^n + (x-1) = (x-1)^n + (x-1)$$

  So, for the cycle graph C_n the chromatic polynomial is $(x-1)^n + (-1)^n(x-1)$.
- The chromatic polynomial of the Petersen graph is $x(x-1)(x-2)(x^7-12x^6+67x^5-230x^4+529x^3-814x^2+775x-352)$ [Refer Wikipedia's page on Petersen graph]

2. This is done by induction. Fix an edge e and let T_e denote the number of triangles that contain edge e. Observe that if G is simple and $f = (x, y)$ is any edge, upon contraction of G by f the only multiple edges in G/f are the ones created by the common neighbours of x, y. So, the number of edges in underlying simple graph of G/e is $|E(G)| - 1 - T_e$ since we have removed one edge and each triangle considered in T_e contributes to a pair of parallel edges. Also, the number of triangles in $G - e$ is $T - T_e$. Let $P_G(x)$ denote the chromatic polynomial of any graph G.
If G is K_2 or a single vertex, the statement is true. Assume that the statement is true for graphs with smaller number of edges. Let G/e denote the underlying simple graph of the contraction of G by e. Then coefficient of x^{n-2} in $P_G(x)$, say α is related to the coefficients of x^{n-2} in $P_{G-e}(x)$, say α_0 and x^{n-2} in $P_{G/e}(x)$, say β_0.
$\beta_0 = -|E| + 1 + T_e$ is the number of edges in G/e. $\alpha_0 = \binom{|E|-1}{2} - (T - T_e)$ by induction hypothesis.
Thus, $\alpha = \binom{|E|-1}{2} - (T - T_e) + |E| - 1 - T_e = \binom{|E|}{2} - T$.
Hence, by induction the statement is true.

3.Induction on n. If $n = 1, 2$, it is clear that the tree's chromatic polynomial is $x(x-1)^{n-1}$. Suppose it is true for any tree on $n-1$ vertices. Let G be a tree on n vertices and v a vertex of degree 1 with e the edge incident on v. Then, $P_G(x) = P_{G-e}(x) - P(G/e)(x)$. G/e is a tree on $n-1$ vertices and $G - e$ has two components, an isolated vertex and a tree on $n-1$ vertices. By induction hypothesis,
$P_G(x) = x^2(x-1)^{n-2} - x(x-1)^{n-2} = x(x-1)^{n-1}$.
Thus, by induction if G is a tree then its chromatic polynomial is $x(x-1)^{n-1}$.
Conversely, suppose G is a graph on n vertices with chromatic polynomial $x(x-1)^{n-1}$. Then G has $n-1$ edges. Since the smallest nonzero coefficient is the coefficient of x, G has only one component, i.e. G is connected. Therefore, G is connected with $n-1$ edges and hence, G is a tree.

Chapter 9

1. Observe that there must be a column with only one entry nonzero for if each column had both $1, -1$, then B would be singular.We may interchange the rows and columns to make this the kth column with the nonzero entry in the kth row. Now, $det(B) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) b_{i\sigma(i)}$, so if a term is nonzero, the summation must be over all σ such that $\sigma(k) = k$, i.e. over all $\sigma \in S_{k-1}$. If $k = 1$, then it is clear that there is only one such permutation. Assume $k \geq 2$, and that the statement is true for $k-1$. Then by the observation above, $det(B) = b_{kk} det(B_{kk})$ where B_{kk} is the minor obtained by deleting the kth column and row. By induction hypothesis, there is only one permutation $\sigma \in S_{k-1}$ such that the product $b_{1\sigma(1)} \dots b_{(k-1)\sigma(k-1)} \neq 0$. This permutation can then be modified to account for the row and column change done earlier. Hence, the statement is true by induction.

2. Let G have n vertices and m edges. Then δ_1 is an $m \times n$ matrix and δ_0 is an $n \times 1$ matrix. δ_i can be seen as linear transformations from C_{i-1} to C_i where $C_{-1} = \mathbb{F}, C_0 = \mathbb{F}^n, C_1 = \mathbb{F}^m$. Since the row sum (sum of entries in a row) of each row of δ_1 is 0, we have $\delta_1 \delta_0 = 0$, hence $Im(\delta_0) \subseteq Ker(\delta_1)$. Since the rank of a matrix and its transpose are the same, the same analysis as with ∂_1, ∂_2 can be used to compute the ranks of image spaces and kernels.

3. If B is invertible, then the columns corresponding to $e_1, \dots e_l$ in B are linearly independent, hence the columns of $e_1, \dots, e_l$ are linearly independent. It follows that $l \leq r(\partial_1)$ and that $e_1, \dots e_l$ form a forest.

4. Some of this (especially those parts regarding adjacency matrices) is taken from the book Graphs and Matrices by R. B. Bapat.
Let $P = v_1 v_2 \dots v_n$ be a path on n vertices and $C = w_1 w_2 \dots w_n w_1$ be a cycle on n vertices. We write the matrices such that the ith row corresponds to the ith vertex. Then

$$A(P) = \begin{bmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & 0 & 1 & 0 \end{bmatrix}, L(P) = \begin{bmatrix} 1 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & 0 & -1 & 1 \end{bmatrix}$$

$$A(C) = \begin{bmatrix} 0 & 1 & 0 & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 1 & & 0 & 1 & 0 \end{bmatrix}, L(C) = \begin{bmatrix} 2 & -1 & 0 & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & 0 & -1 & 2 \end{bmatrix}$$

Observe that the first $k \times k$ minor of $A(C)$ is the adjacency matrix of a path graph on k vertices for $k < n$.

First we deal with cycles. Let $D = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{bmatrix}$ be the permutation matrix that maps the vector $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ to $\begin{bmatrix} x_2 & x_3 & \dots & x_1 \end{bmatrix}^T$. If x is an eigenvector with eigenvalue λ, then $x_2 = \lambda x_1, x_{i+1} = \lambda x_i$, so recursively, we get $x_i = \lambda^{i-1} x_1$ and thus, $x = x_1 \begin{bmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^{n-1} \end{bmatrix}$.
Hence, $x_1 \neq 0$ and λ is an nth root of unity. This also shows that all nth roots of unity are eigenvalues of D.
Now, $A(C) = D + D^T$ and $D^T = D^{n-1}$. Thus, the eigenvalues of $A(C)$ are $\lambda + \lambda^{n-1}$ where λ is an nth root of unity. Using $e^{i\theta} = \cos(\theta) + i \sin(\theta)$, we have the eigenvalues of $A(C)$ to be $\{2\cos(2\pi i/n) \colon 0 \leq i \leq n-1\}$.
Since $L(C) = 2I_n - A(C)$, the eigenvalues of $L(C)$ are $\{4\sin^2(\pi i/n) \colon 0 \leq i \leq n-1\}$

Now, if $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ is an eigenvector of $A(P)$ with eigenvalue λ, then by symmetry $x' = \begin{bmatrix} -x_n & -x_{n-1} & \dots & -x_1 \end{bmatrix}^T$ is also an eigenvector with the same eigenvalue.
Let C' be the cycle on $2n+2$ vertices, then

$$A(C') = \left[\begin{array}{c:c:c:c} A(P) & e_n & 0_{n\times n} & e_1 \\ \hdashline e_n^T & 0 & e_1^T & 0 \\ \hdashline 0_{n\times n} & e_1 & A(P) & e_n \\ \hdashline e_1^T & 0 & e_n^T & 0 \end{array}\right]$$

where $e_1, e_2, \dots, e_n$ are the standard (column) basis vectors.
Now, observe that $\begin{bmatrix} x & 0 & x' & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & x & 0 & x' \end{bmatrix}$ are two independent eigenvectors of $A(C')$ both having eigenvalue

λ.
Now, the eigenvalues of C' are $2\cos(\frac{2\pi i}{2n+2}) = 2\cos(\frac{\pi i}{n+1})$ for $i = 0, 1, 2, \ldots, 2n+1$. Of these, the eigenvalues that have multiplicity 2 are $2\cos(\frac{\pi i}{n+1})$ for $i = 1, \ldots, n$. Since none of these values occur more than two times, no eigenvalue of P has multiplicity greater than 1 (as then C' would have an eigenvalue of multiplicity ≥ 4). Thus, these must be all the eigenvalues of P.

Let $x = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$ be a column vector. By the reverse of x we mean the vector $x^r = \begin{bmatrix} x_n & x_{n-1} & \ldots & x_1 \end{bmatrix}$. Now if x is an eigenvector of $L(P)$, then observer that $\begin{bmatrix} x & x^r \end{bmatrix}^T$ is an eigenvector of $L(C_{2n})$ with the same eigenvalue.
Now if we assume the vector $y = \begin{bmatrix} x & x^r \end{bmatrix}$ is an eigenvector of $L(C_{2n})$ with eigenvalue λ, then x is an eigenvector of $L(P)$ with the same eigenvalue.

From the first equation of $L(C_{2n})y = \lambda y$, we get $x_2 = (1-\lambda)x_1$. Define $P_2(x)$ to be the polynomial $1 - x$. From the second equation we get $x_3 = P_3(\lambda)x_1$ where $P_3(x)$ is a quadratic in x. Continuing this way, since all the $i-1$th equation is linear in x_i, we get $x_i = P_i(\lambda)x_1$ where $P_i(x)$ is a degree $i-1$ polynomial in x for $1 \leq i \leq n$. So, each x_i is a multiple of x_1, or in other words the eigenspace corresponding to λ has dimension 1 and because $y \neq 0$, we must have $x_1 \neq 0$. Thus, from the nth equation, we see that λ must satisfy the polnomial $(1-x)P_n(x) - P_{n-1}(x) = 0$. The rest of the equations, i.e. from the $(n+1)$st to the $2n$th equation, are the reverse of the first n equations by the symmetry of $L(C_{2n})$ and y.
Now, we can go in the other direction and it is clear that if λ is a root of the degree n polynomial $(1-x)P_n(x) - P_{n-1}(x)$, then it is an eigenvalue of $L(C_{2n})$ by the way we constructed the polynomials P_i.

So far, we have deduced that λ is an eigenvalue of $L(P)$ if and only if it is a root of $Q(x) = (1-x)P_n(x) - P_{n-1}(x)$ and that each root of $Q(x)$ is an eigenvalue of $L(C_{2n})$, so $Q(x)$ has real roots only.
Now, if x is an eigenvalue of $L(P)$, we can, as done above, see that each x_i must be a multiple of x_1, i.e. the eigenspaces of $L(P)$ are all 1 dimensional. It follows that $Q(x)$ is separable (has distinct roots) and is a constant multiple of the minimal and characteristic polynomials of $L(P)$.

$L(C_{2n})$ has $n+1$ distinct eigenvalues of which 0, 4 have multiplicity 1 and the rest have multiplicity 2. We know that 0 is an eigenvalue of $L(P)$. That leaves us with $n-1$ remaining eigenvalues. So, one of the remaining n eigenvalues of $L(C_{2n})$ is not an eigenvalue of $L(P)$.
Finally, observe that 4 is not an eigenvalue of $L(P)$ because the spectrum of $L(P)$ is symmetric as P is bipartite and $2 - 2\cos(x)$ cannot take the value -4. Thus, apart from 4 all the other eigenvalues of $L(C_{2n})$ are eigenvalues of $L(P)$.

5. Let A be the adjacency matrix of a graph on n vertices $v_1, \ldots, v_n$ having e edges. Let A have eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.
(a)Note that $Tr(A) = \sum_{i=1}^{i=n} \lambda_i = 0$ and $Tr(A^2) = \sum_{i=1}^{i=n} \lambda_i^2 = \sum_{i=1}^{i=n} d(v_i) = 2m$. So, we have

$$\begin{aligned} \lambda_1 &= -(\lambda_2 + \cdots + \lambda_n) \\ \Rightarrow |\lambda_1| &= |\lambda_2 + \cdots + \lambda_n| \\ &\leq \sqrt{\lambda_2^2 + \ldots \lambda_n^2}\sqrt{n-1} \\ &= \sqrt{2m - \lambda_1^2}\sqrt{n-1} \\ \Rightarrow \lambda_1^2 &\leq 2m(n-1) - \lambda_1^2(n-1) \\ \Rightarrow \lambda_1 &\leq \sqrt{\frac{2m(n-1)}{n}} \end{aligned}$$

(b) Let λ be any eigenvalue of A, and $X = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$ a corresponding eigenvector. Then $AX = \lambda X$. For each row j of A, we have

$$\sum_{i \sim j} x_i = \lambda x_j$$

Now let k be such that $|x_k|$ is largest. If necessary, use $-X$, so that x_k is nonnegative. Then,

$$\lambda x_k = \sum_{i \sim k} x_i \leq \sum_{i \sim k} x_k = d(v_k)x_k \leq \Delta(G)x_k$$

Since $X \neq 0, x_k > 0 \Rightarrow \lambda \leq \Delta(G)$. So, any eigenvalue is less than $\Delta(G)$, in particular $\lambda_1 \leq \Delta(G)$.

Lemma : $\lambda_1 = \max\limits_{|X|=1} X^TAX$

Proof : Since A is symmetric, there is an orthogonal matrix P such that $PAP^T = D$ where $D = \text{diag}(\lambda_1 \dots \lambda_n)$. Then, $X^TAX = X^TP^TDPX = (PX)^TD(PX)$.
Now, since P is invertible, we have $\max\limits_{|X|=1} X^TAX = \max\limits_{|X|=1} X^TDX$. Now, D being diagonal, $X^TDX = \lambda_1 x_1^2 + \dots \lambda_n x_n^2 \leq \lambda_1(x_1^2 + \dots + x_n^2) = \lambda_1$. Further, for $X = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}, X^TDX = \lambda_1$. This proves the lemma. ■

Observe that a similar proof shows that $\lambda_n = \min\limits_{|X|=1} X^TAX$.

Now, take $X = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \dots & \frac{1}{\sqrt{n}} \end{bmatrix}^T$, then $X^TAX = \frac{2m}{n} = d(G)$. So, $\lambda_1 \geq d(G) \geq \delta(G)$, where $d(G)$ is the average degree.

(c) Let H be the induced subgraph on vertices$\{v_1, \dots, v_p\}$. Now, if x is a p-vector, then let x' be the n-vector obtained by padding zeros. Let A be such that the first p-rows are indexed by $\{v_1, \dots, v_p\}$, then the adjacency matrix of H is the first $p \times p$ minor of A, A_p. Observe that $x^TA_px = x'^TAx'$. Then it follows that

$$\lambda_n(G) = \min_{|X|=1} X^TAX \leq \lambda_p(H) = \min_{|X|=1} X^TA_pX \leq \lambda_1(H) = \max_{|X|=1} X^TA_pX \leq \lambda_1(G) = \max_{|X|=1} X^TAX$$

where the maxima and minima are taken over appropriate spaces.

6. Let $P = w_0w_1 \dots w_k$ be a path from v_j to v_i in G. Such a path exists since G is connected. Construct the vector $X = \begin{bmatrix} x_1 & x_2 & \dots & x_m \end{bmatrix}^T$ by taking $x_i = \mathbf{1}[x_i \in P] - \mathbf{1}[-x_i \in P]$, i.e. 1 if P goes along edge i, -1 if it goes against edge i and 0 if P doesn't use edge i.
For a vertex v, let R_v be the row of ∂_1 corresponding to v.
If $v \notin P$, then for each edge e incident on v, $x_e = 0$, so $R_vX = 0$.
If $v = v_i$, then there is only one edge e in P which is incident on v_i, so, $R_vX = \partial_1(v_i, e)x_e = 1$.
If $v = v_j$, then similar to $v_i, R_vX = 1$.
If $v \in P$ and $v \neq v_i, v_j$ then there are two edges of P incident on v. By construction of X, it is clear that $R_vX = 0$.
Thus X is the required vector. From a graph theoretical point of view, the vector X corresponds to that sequence of directed edges in G that lets us go from v_j to v_i in a directed manner similar to the vector y which is a directed edge from v_j to v_i. More elaborately, if the directed edges of G are thought of as vectors in $\mathbb{R}^2$, then X is the vector that gives y as sums of displacement vectors given by the edges.
Note that it is necessary that G be connected. Here is a counter example when G is not connected :

$\partial_1 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ and $y = \begin{bmatrix} 0 & 0 & -1 & 0 & 1 \end{bmatrix}^T$

$\partial_1 x = y$ has no solution.

7.We know that $x = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$ is an eigenvector of J with eigenvalue 0. Starting with $\frac{1}{\sqrt{n}}x$ as the first column, we can construct an orthogonal matrix P such that $P^TLP = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ where $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ are the eigenvalues of L.
Since P is orthogonal and the first column is $\frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$, it follows that the other columns are orthogonal to $\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$. Thus,

$$P^TJ = \begin{bmatrix} \sqrt{n} & \sqrt{n} & \dots & \sqrt{n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \Rightarrow P^TJP = \begin{bmatrix} n & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}$$

Thus, $P^T(L+aJ)P = \begin{bmatrix} na & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$. It follows that the eigenvalues of $L+aJ$ are $na, \lambda_2, \ldots, \lambda_n$.

8. Since the Petersen graph G is 3-regular the diagonal entries of A^2 are all 3. By Exercise 3.2.12, two non adjacent vertices of G have exactly one common neighbour, so $A^2(i,j) = 1$ if $i \not\sim j$. Further, if $i \sim j$, then the number of walks of length 2 from i to j is 0, so $A^2(i,j) = 0$ if $i \sim j$. Thus, $A^2 + A = 2I + J$ or $A^2 + A - 2I = J$.
Now, J has eigenvalues 0 with multiplicity $n-1 = 9$ and $n = 10$ with multiplicity 1. If λ is an eigenvalue of A, then $\lambda^2 + \lambda - 2$ is an eigenvalue of J, so λ must satisfy $\lambda^2 + \lambda - 2 = 10$ or $\lambda^2 + \lambda - 2 = 0$. The solutions of the first equation are $-4, 3$ and the solutions to the second equation are $-2, 1$. Of the first two, both can't be eigenvalues of A as then 10 would appear with multiplicity 2 for J because eigenvectors of different eigenvalues are independent. Observe that $Ax = 3x$ where $x = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T$. So -4 is not an eigenvalue of A.
Let -2 appear as an eigenvalue of A with multiplicity a and 1 with multiplicity b, then $a + b = 9$ and since $Tr(A) = 0$, we also have $3 + b - 2a = 0$. Solving these gives $a = 4, b = 5$. Since the solution for the above system of equations are unique, $-2, 1$ must be eigenvalues of A with the said multiplicities. Thus the eigenvalues of A are $-2, -2, -2, -2, 1, 1, 1, 1, 1, 3$. Since G is 3-regular, $L = 3I - A$, so the eigenvalues of L are $0, 2, 2, 2, 2, 2, 5, 5, 5, 5$.

9. Let G, H be graphs with n, m vertices respectively. Then, $G \times H$ has nm vertices. Let the vertices of H be $v_1, \ldots, v_m$ with degrees $d_1, \ldots, d_m$ respectively. There are m copies of G in $G \times H, G_1, \ldots G_m$ with G_i occupying the vertex v_i of H. Let the Laplacian of G be $L(G)$, Laplacian of H be $L(H)$.

We can order the vertices of $G \times H$ in such a way that the Laplacian of $G \times H$ has the form $\begin{bmatrix} B_{11} & B_{12} & \ldots & B_{1m} \\ B_{21} & B_{22} & \ldots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \ldots & B_{mm} \end{bmatrix}$ where each B_{ij} is an $n \times n$ sqaure matrix such that the rows $\begin{bmatrix} B_{i1} & B_{i2} & \ldots & B_{im} \end{bmatrix}$ corresponds to the copy G_i of G in $G \times H$.
Then observe that $B_{ii} = L(G) + d_i I$, and $B_{ij} = -I\mathbf{1}[L(H)_{ij} = -1]$ for $i \neq j$, where I is the $n \times n$ identity matrix.
This has a particulary nice closed form : $L(G \times H) = L(G) \otimes I_m + I_n \otimes L(H)$.
Now, let the eigenvalues of G be $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n = 0$ and that of H be $\mu_1 \geq \mu_2 \geq \mu_m = 0$. Let x be an eigenvector of $L(G)$ with eigenvalue λ_i and y be an eigenvector of $L(H)$ with eigenvalue μ_j
With the representation of $L(G \times H)$ as above, it is easy to see that $x \otimes y$ (mn column vector) is an eigenvector of $L(G \times H)$ with eigenvalue $\lambda_i + \mu_j$. Further, if $x_1, \ldots, x_t$ are linearly independent eigenvectors of $L(G)$ with eigenvalue λ_i and $y_1, \ldots, y_k$ are linearly independent eigenvectors of $L(H)$ with eigenalue μ_j, then $\{x_r \otimes y_s \colon 1 \leq r \leq t, 1 \leq s \leq k\}$ are linearly independent. Then, by looking at the dimensions of eigenspaces, we conclude that the eigenvalues of $L(G \times H)$ are $\{\lambda_i + \mu_j \colon 1 \leq i \leq n, 1 \leq j \leq m\}$.

Chapter 10

Exercise 10.0.4

If e is not a cut edge, then it occurs in the boundary of exactly two faces and hence is counted twice. If e is a cut edge, then it occurs in the boundary of exactly one face, but by definition of $l(F)$, it is still counted twice. Thuse each edge is counted twice and hence $\sum_F l(f) = 2|E|$.

Exercise 10.1.5

If G is a triangulation, then $e = 3v - 6$ and adding any more edges violates this equality rendering G nonplanar. Thus, G is maximal planar.
Next, suppose G is not a triangulation, then there is a face F with length ≥ 4. If G is not connected, then we can add edges between components to get larger planar graphs, so assume that G is connected. If the boundary of F is acyclic, then its boundary is a tree on at least 3 vertices since G is connected. We may add edges in non adjacent vertices and get larger planar graphs. So, let its boundary contain cycles. If there is a cut edge involved, then there are leaf vertices in the boundary of F for the cut edges form a forest. We may add an edge from this leaf vertex to any vertex in the cycle increasing the number of edges while keeping it planar. Finally, if the boundary of F is a cycle, then since it has length ≥ 4, we can add an edge (e.g. a diameter) giving a larger planar graph. Thus, if G is not a triangulation, then it is not maximal planar.

Exercise 10.3.1

For a face F, $d_{G^*}(F) = l(F)$ as each edge of a cycle contributes one neighbouring face, and each cut edge in ∂F contributes a loop, increasing the degree by 2.
(1)$\Rightarrow$(2): If G is bipartite, then all cycles are even. So, by definition of $l(F), l(F)$ is even for every face F.
(2)$\Rightarrow$(3): Note that the dual of a planar graph is always connected by the external face [see Exercise 1 of Additional exercises]. So the dual graph G^* of G is connected. Further, in G^*, the degree of a vertex is the length of the corresponding face F. Since $l(F)$ is even for each F, G^* is Eulerian.
(3)$\Rightarrow$(1): If G^* is Eulerian, then there is no odd length face in $G \Rightarrow$ no odd cycles. Thus G is bipartite.

Exercise 10.4

1. Given two edges e_1, e_2 in a graph G, call them type 0 if they are disjoint, type 1 if they have 1 vertex in common and type 2 if they are parallel, i.e. have the same end vertices. Let G be planar, embed it into $\mathbb{R}^2$ such that no two edges cross. Contract an edge e. Observe that upon contraction of e, two different edges $e_1, e_2 \neq e$ could go from being type 2 to being type 2 or type 1 and if they started out as type 1, they could go from type 1 to type 1 or type 0. In all possible cases they do not cross each other in G/e. Further, if an edge f was parallel with e, then upon contraction f becomes a loop, and since there is a finite number of edges, this can be made small enough so as to not cross any other edge.
Thus, two edges that don't cross in G dont't cross in G/e as well. Hence, G/e is planar.

2. Euler's formula is true for any embedding of a planar graph in $\mathbb{R}^2$. Since any embedding preserves n, e, it follows that f is the same for every embedding.

3. Assume $n \geq 3$ as this is true for simple graphs on 2 vertices. If all vertices had degree ≥ 4, then $2e \geq 4n$ but since G is triangle free, $2e \leq 4n - 8$. This is a contradiction, so there is at least one vertex of degree 3 or less. In fact, there have to be at least 3 vertices of degree 3 or less : let a be the number of vertices of degree 3 or less, then if G is connected, $2e \geq 4(n-a) + a$ as each of those a vertices have degree ≥ 1. Then $2n - 4 \geq 2n - \frac{3a}{2}$ give $a \geq \frac{8}{3}$ So, each connected component has at least 3 vertices of degree ≤ 3, and hence the same holds for G.
Now, if $n = 3, 4$, then G is 4-colourable. Assume any planar triangle free graph on $< n$ vertices is 4-colourable. Let x be a vertex of G with degree ≤ 3, then each component of $G - x$ having fewer vertices and being triangle free planar graphs is 4-colourable. Colour $G - x$ with 4 colours. Now, x has at most 3 neighbours, so one of the four colours is not used by the neighbours of x, use this colour for x. The resulting colouring is a proper colouring of G using 4-colours.

4. Consider a maximal planar graph G' containing G as a subgraph. If G' has at least 4 vertices of degree at most 5, then it follows that G also has at least 4 vertices of degree 5. Now G' is connected. Since G' is maximal planar, it is a triangulation and satisfies $3n - 6 = e$. If v is a vertex of degree ≤ 2, then $G - v$ is maximal planar on $n - 1$ vertices with $e - d(v)$ edges. But $d(v) \leq 2$ gives $3n - 9 > e - d(v)$ contradicting maximality of $G' - v$. Thus, each vertex of G' has degree ≥ 3. Let a vertices have degree $3, 4$ or 5, and let d be the sum of their degree. The remaining $n - a$ vertices have degree ≥ 6. We have the inequalities

$$3n - 6 \geq e \geq 3(n-a) + d/2 \geq 3(n-a) + 3a/2$$

This gives $a \geq 4$ as required.

Additional exercises

1.Let G be a planar graph and G^* its dual. We shall induct on the number of edges of G and show that there is a path from every face to the external face. If G has only one edge, then it is clear that G^* having only one vertex is connected. Now, suppose that G^* for every graph G with lesser number of edges. Let F be the external face of G. If F has a cut edge in its boundary, we may remove that edge as that adds a loop at F, and continue with the induction. So, assume F has no cut edge in its boundary. Let $e \in \partial F$, then e is part of some cycle. This cycle encloses a face $F' \neq F$, and the dual of e provides the path from F' to F. Now, remove e. Then, F' merges with F and all other faces remain unaltered. This new graph has fewer edges and induction hypothesis applies. Thus, every face in this new graph has a path to the external face of this new graph. Going back to G, it is clear that every face has a path to F or F'. Since there is already a path from F' to F, it follows that every face has a path to F. Thus, G' is connected.
Now, let v be a non isolated vertex with edges $e_1, \dots e_l$ incident on v. Now, between consecutive edges, there is a face of G, i.e. a vertex of G^*, these may not be distinct. However, these faces form a cycle around v, where some of the vertices may be repeated. Now, if these faces are all the same, then we have l loops around it, and we can use the external face to enclose v, else, we have at least two vertices of G^* and the edges between them enclose v. Observe that in doing so, we have cut v off from all other vertices that are reachable from v, i.e. those vertices that are reachable from v cannot be in the same face as v.
So, if G is connected, then each vertex of G is contained in one face of G^* and that face contains only v and no other vertices of G. Since the number of edges in G and G^* are the same, from Euler's formula, each face of G^* contains exactly one vertex of G.
Finally, if G is connected, each face of G^* contains exactly one vertex of G and each vertex of G is contained in a face of G^*. Further, by construction of duals, it is clear that the degree of a face of G^* is the same as the degree of the corresponding vertex of G. This gives the required equality.

2.Note that since G is connected $(G^*)^* = G$, so it is enough to prove one direction. Let T be a spanning tree of G, and T^* the dual edges corresponding to edges of $E(G) \setminus T$. We have $|T^*| = e - (n-1) = f - 1$ and G^* has f vertices. So it is enough to prove that the subgraph of G^* generated by T^* is acyclic or connected.
Let H be the subgraph generated by $E(G) \setminus T$, then H^* is the subgraph of G^* generated by T^*. Since the dual of any graph is connected, H^* is connected. Since T^* has the right cardinality, it follows that T^* is a spanning tree of G^*.
Alternatively, if T^* has a cycle C, then by embedding G^* in $\mathbb{R}^2$, C contains a face of G^* and so does the complement of C in $\mathbb{R}^2$. These faces contain vertices of G and C separates them, i.e. every path connecting these vertices should pass through $E(G) \setminus T$. Thus, $E(G) \setminus T$ contains a cut set which is a contradiction as T is connected.

3. If G is self-dual, then $f = n$ and by Euler's formula, $e = 2n - 2$. Consider the wheel graph W_n on $n+1$ vertices for $n \geq 3$. This is a self dual graph because the external face has degree n, the same as inner centre vertex and each of the other n faces have degree 3, which can be matched with the degree 3 outer vertices.