On completions of metric spaces

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Let (X, d) be a metric space. We say that (X', d') is a completion of X if it is a complete metric space and there is an isometry $\phi\colon X \to X'$ such that $\phi(X)$ is dense in X'.

Theorem 1. (*Existence of completions*) *Every metric space has a completion.*

Proof. Let (X, d) be a metric space and let C be the set of all Cauchy sequences in X. One can consider this as a subset of all functions $\mathbb{N} \to X$ determined by the Cauchy condition. Define a relation on C as follows. Two sequences $(x_n), (y_n)$ are said to be related if the sequence $d(x_n, y_n)$ converges to 0. One can easily check that this is an equivalence relation, let $\tilde{X}$ be the collection of all equivalence classes, and denote by $[x_n]$ the class of the sequence (x_n).

Now, define a metric on $\tilde{X}$ as

$$\begin{aligned} d'\colon \tilde{X} \times \tilde{X} &\to \mathbb{R} \\ ([x_n], [y_n]) &\mapsto \lim_{n\to\infty} d(x_n, y_n) \end{aligned}$$

We need to verify that d' is well defined and indeed defines a metric on $\tilde{X}$.

1. Let $(x_n), (y_n)$ be two Cauchy sequences in X, then the sequence $d(x_n, y_n)$ is Cauchy in $\mathbb{R}$: Given $\epsilon > 0$, choose N such that for $m, n > N, d(x_n, x_m) < \epsilon, d(y_n, y_m) < \epsilon$. Then,

$$d(x_m, y_m) \le d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m)$$

so we conclude that $|d(x_m, y_m) - d(x_n, y_n)| < 2\epsilon$. Therefore, the limit exists.

2. Suppose $(x_n), (y_n)$ are related to $(x'_n), (y'_n)$ respectively. Then

$$d(x'_n, y'_n) \le d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n)$$

and taking limits (which is allowed because all limits above exist) we get

$$\lim d(x'_n, y'_n) \le \lim d(x_n, y_n).$$

Similarly, $\lim d(x_n, y_n) \le \lim d(x'_n, y'_n)$. Therefore, d' is a well defined map.

3. Clearly d' is nonnegative and symmetric. Suppose $[x_n], [y_n], [z_n] \in \tilde{X}$, then $d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n)$. Taking limits, it follows that d' is a metric on $\tilde{X}$.

Next we need to embed X in $\tilde{X}$. We accomplish this via constant sequences, define

$$\begin{aligned} \phi\colon X &\to \tilde{X} \\ x &\mapsto [x] \end{aligned}$$

where (x) is the constant sequence $(x, x, \ldots,)$ (which is obviously a Cauchy sequence). It is easy to verify that ϕ is an isometry. Given any $[x_n] \in \tilde{X}$, we see that the sequnce $\phi(x_n)$ converges to $[x_n]$ because (x_n) is a Cauchy sequence. Therefore, $\phi(X)$ is dense in $\tilde{X}$.

Lastly we need to verify that $\tilde{X}$ is a complete metric space. To prove this, we use the fact that $\phi(X)$ is dense in $\tilde{X}$. Let $\{[x_{nm}]\}_{m\geq 1}$ be a Cauchy sequence in $\tilde{X}$ and for each m, choose a $y_m \in X$ such that

$$d'(\phi(y_m), [x_{nm}]) < 1/m.$$

We claim that $\{[x_{nm}]\}_{m\geq 1}$ converges to the class y of $(y_1, y_2, \dots,)$. Firstly note that the sequence $(y_1, y_2, \dots,)$ is Cauchy in X because

$$d(y_r, y_s) = d'(\phi(y_r), \phi(y_s)) \leq d'(\phi(y_r), [x_{nr}]) + d'([x_{nr}], [x_{ns}]) + d'([x_{ns}], \phi(y_s))$$

and the right side can be made small (note that the sequence $1/k$ is decreasing).

Therefore, the Cauchy sequence $\phi(y_r)$ converges in $\tilde{X}$ to y. By construction, $\{[x_{nm}]\}_{m\geq 1}$ also converges to y. Thus, $(\tilde{X}, d')$ is a completion of X. □

Remark. Above we considered a particular equivalence relation on the space of Cauchy sequences and considered the equivalence classes. In general, given a topological space, we can identify points that are "close" to each other as an equivalence relation and taking its space of equivalence classes as a quotient topology, we can get a Hausdorff space. This procedure is called *Hausdorffification* and has certain nice properties: it is the adjoint of the inclusion (forgetful) functor from the category of Hausdorff spaces to the category of topological spaces.

Definition 1. *Let $(X, d), (Y, d')$ be metric spaces. A map $f\colon X \to Y$ is said to be Lipshcitz continuous with Lipschitz constant k if $\forall x, y \in X, d'(f(x), f(y)) \leq kd(x, y)$. A Lipschitz map is always continuous.*

In particular, isometries are Lipschitz continuous.

Theorem 2. *Suppose $X \subset Y$ with Y a metric space and let Z be a complete metric space. Suppose $f\colon X \to Z$ is Lipschitz continuous with constant k, then there is a unique continuous extension $\bar{f}\colon \bar{X} \to Z$ of f. Moreover, $\bar{f}$ is also Lipschitz with the same constant. If f is an isometry, then so is $\bar{f}$.*

Proof. As far as uniqueness goes, we only need f to be continuous. We also need the following easy fact : when X is first countable and Z is any space, a function $f\colon X \to Z$ is continuous if and only if whenever $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Z. Moreover, in metric spaces, $x \in \bar{X}$ if and only if there is a sequence in X converging to $\bar{X}$.

So, if $f_1, f_2\colon \bar{X} \to Z$ are two continuous extensions of f, given $x \in \bar{X}$, choose a sequence x_n in X converging to x. Then, $f_1(x) = \lim f(x_n) = f_2(x)$. To prove that extensions exist, we will need all the additional hypotheses.

Let d be the metric on Y, d' that on Z. Define

$$\begin{aligned}\bar{f}\colon \bar{X} &\to Z\\ x &\mapsto \lim f(x_n) : \{x_n\}_{n\geq 1} \subseteq X, x_n \to x\end{aligned}$$

i.e., we take $\bar{f}(x)$ to be the limit of the sequence $f(x_n)$ when x_n is a sequence in X converging to x. Firstly, because $x_n \to x$, it is a Cauchy sequence and since f is Lipschitz, $\lim f(x_n)$ exists in Z. Next, if we have two sequences $(x_n), (x'_n)$ converging to x, then $d(x_n, x'_n) \to 0$, hence $d'(f(x_n), f(x'_n)) \to 0$. This means that the two sequences $f(x_n), f(x'_n)$ have the same limits. Therefore, $\bar{f}$ is a well defined map, and is clearly an extension of f (by considering constant sequences).

All that is left to show is that $\bar{f}$ is Lipschitz. Let $x, y \in \bar{f}$ and let $x_n \to x, y_n \to y$. Then $f(x_n), f(y_n)$ converge to $\bar{f}(x), \bar{f}(y)$ respectively. A simple calculation shows that $d(x_n, y_n)$ converges to $d(x, y)$ and that $d'(f(x_n), f(y_n))$ converges to $d'(\bar{f}(x), \bar{f}(y))$. Taking the limits of both sides of

$$d'(f(x_n), f(y_n)) \leq kd(x, y)$$

gives us $d'(\bar{f}(x), \bar{f}(y)) \leq d(x, y)$ as required. If f is an isometry, then it follows similarly that $\bar{f}$ is also an isometry. □

Corollary 1. *Let $(X_i, d_i), i = 1, 2, 3$ be metric spaces, $(\bar{X}_i, \bar{d}_i)$ be their respective completions with isometries $\phi_i\colon X_i \to \bar{X}_i, i = 1, 2, 3$. Any Lipschitz continuous $f\colon X_1 \to X_2$ can be lifted to $\bar{f}\colon \bar{X}_1 \to \bar{X}_2$ such that $\bar{f} \circ \phi_1 = \phi_2 \circ f$. Moreover, if $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$ are both Lipschitz, then the lift of $g \circ f$ (which is also Lipschitz continuous) is the composition $\bar{g} \circ \bar{f}$.*

Proof. Simply consider the Lipschitz function $\phi_2 \circ f\colon X_1 \to \bar{X}_2$ and apply the theorem. The second fact follows from the uniquenss of the extension. Note that the corresponding statements hold if one replaces Lipschitz with isometries. $\square$

Corollary 2. *Completion of a metric space is uniquely determined upto homeomorphism. In fact, let (X, d) be a metric space, $(X_1, d_1), (X_2, d_2)$ be completions of X with isometries $\phi_i\colon X \to X_i, i = 1, 2$, then there is a unique surjective isometry $\phi\colon X_1 \to X_2$ such that $\phi \circ \phi_1 = \phi_2$.*

Proof. Because ϕ_1 is an isometry, $\phi_1\colon X \to \phi_1(X)$ is a homeomorphism. Therefore, we have an isometry $\phi_2 \circ \phi_1^{-1}\colon \phi_1(X) \to X_2$. By the theorem above, this extends to a unique isometry $\phi\colon X_1 \to X_2$. Similarly, we have a unique isometry $\psi\colon X_2 \to X_1$ which is the extension of $\phi_1 \circ \phi_2^{-1}\colon \phi_2(X) \to X_1$. Then, we see that $\phi \circ \psi\colon X_2 \to X_2$ is an isometric extension of the inclusion map $\phi_2(X) \hookrightarrow X_2$, hence must be the identity map by uniqueness. Similarly, $\psi \circ \phi = id_{X_1}$. Clearly $\phi \circ \phi_1 = \phi_2$. $\square$

As a consequence, if M be the category of metric spaces with isometries/Lipschitz continuous functions as the corresponding homomorphisms. Then, because the completion is uniquely determined upto isometry, taking the completion is a functor from M to M. Moreover, the unique extension property takes commutative diagrams to commutative diagrams, hence it is possible to lift most algebraic structures, vector space structure, inner products etc., from a metric space X to its completion.

Lemma 1. *1. Let $\{X_\alpha\}_{\alpha \in I}$ be topological spaces and let $X = \Pi_{\alpha \in I} X_\alpha$ be the product topology. A sequence $\{(x_\alpha)_n\}$ converges to (a_α) in X if and only if each of the coordinates converge to a_α in X_α.*
2. Let $(X_1, d_1), (X_2, d_2)$ be metric spaces. Then $X_1 \times X_2$ is a metric space.
3. Let $(X_n, d_n)_{n \geq 1}$ be bounded metric spaces, then $\Pi_{n \geq 1} X_n$ is also a metric space.

Proof. 1. Suppose $\{(x_\alpha)_n\}$ converges to (a_α). Let U_α be a neighbourhood of a_α, then by considering $\pi_\alpha^{-1}(U_\alpha)$, where $\pi_\alpha\colon X \to X_\alpha$ is the projection map, it follows that some tail of the sequence $\pi_\alpha((x_\alpha)_n$ lies in U_α. This way, the αth coordinate converges to a_α. This also follows more directly from the fact that the projection map is continuous. Conversely, suppose the αth coordinate converges to a_α, then consider basic open sets around (a_α), it follows that the sequence converges to (a_α).

2. On $X_1 \times X_2$ define

$$d((x_1, x_2), (y_1, y_2)) = (d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2)^{1/2}.$$

It is easily verified that d is a metric. Fix some $p = (x_0, y_0) \in X_1 \times X_2$. The ball $B_d(p_0, \epsilon)$ in the metric d contains the open set $B_{d_1}(x_0, \epsilon/\sqrt{2}) \times B_{d_2}(y_0, \epsilon/\sqrt{2})$. And this open ball in turn contains the ball $B_d(p_0, \epsilon/\sqrt{2})$. It follows that the product topology on $X_1 \times X_2$ is the same as the one induced by d.

3. A metric d on X is bounded by c if for every $x, y \in X, d(x, y) \leq c$. Now, if d is a bounded metric, then we may assume that it is bounded by 1 for scaling the metric doesn't change the topology. So, assume that each d_n is bounded by 1 and consider the metric d on $X = \Pi_{n \geq 1} X_n$ given by

$$d((x_n), (y_n)) = \sum_{n \geq 1} \frac{d_n(x_n, y_n)}{2^n}.$$

It is easy to verify that d is indeed a well defined metric (while proving triangle inequality, keep in mind that everything is absolutely convergent, so one can rearrange and partition the series). Fix a sequence $p = (x_n)$ and consider the open ball $B_d(p, \epsilon)$. Choose an N such that $\sum_{n > N} 1/2^n < \epsilon/2$. Then the ball $B_d(p, \epsilon)$ contains the basic open set $B_{d_1}(x_1, \epsilon/N) \times B_{d_2}(x_2, 2\epsilon/N) \times \cdots \times B_{d_N}(x_N, 2^{N-1}\epsilon/N) \times X_{N+1} \times \ldots$ which in turn contains the open ball $B_d(p, \epsilon/2N)$. Therefore, the product topology is the same as that induced by d. $\square$

Note that in the metric space $X_1 \times X_2$ a sequence is Cauchy if and only if each of its coordinate sequences are Cauchy. Moreover, the same procedure extends directly to a finite product of metric spaces. Combining the first two parts of the lemma, we see that if $(X_1, d_1), (X_2, d_2)$ are both complete, then so is $X_1 \times X_2$. As a consequence, the metric completion of a finite product of metric spaces is the product of their individual completions (with the sum of squares type distance).

We move on to Hilbert spaces. Let F be a field, which may be $\mathbb{R}$ or $\mathbb{C}$, and $\mathscr{X}$ a vector space over F. An inner product on $\mathscr{X}$ is a map

$$\langle \cdot, \cdot \rangle \colon \mathscr{X} \times \mathscr{X} \to F$$

such that

1. It is linear in the first variable, conjugate linear in the second.
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x, x \rangle \geq 0$
4. $\langle x, x \rangle = 0 \implies x = 0$ (without this condition we have a semi-inner product)

By linearity and conjugate linearity, we have $\langle x, 0 \rangle = \langle 0, x \rangle = \forall x \in \mathscr{X}$. A norm on $\mathscr{X}$ is a map

$$\|\cdot\| : \mathscr{X} \to \mathbb{R}$$

such that

1. $\|\alpha x\| = |\alpha| \, \|x\| \, \forall x \in \mathscr{X}, \alpha \in F$
2. $\|x + y\| \leq \|x\| + \|y\|$
3. $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.

From an inner product we obtain the norm defined as $\|x\| = \langle x, x \rangle^{1/2}$.

Theorem 3. (*Cauchy-Bunyakowsky-Schwartz inequality*) *If $\langle \cdot, \cdot \rangle$ is an inner product on $\mathscr{X}$, then for any $x, y \in \mathscr{X}$,*

$$|\langle x, x \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

and equality holds if and only if there exist $\alpha, \beta \in F$ both not equal to zero such that $\langle \beta x + \alpha y, \beta x, \alpha y \rangle = 0$.

Proof. See [1]. $\square$

Definition 2. *A Hilbert space is a vector space $\mathscr{X}$ with an inner product $\langle \cdot, \cdot \rangle$ which is complete with respect to the induced norm.*

Theorem 4. *Every inner product space has a completion which is a Hilbert space.*

Proof. Let $(\mathscr{X}, d)$ be an inner product space (d being induced by the inner product) and let $(\bar{\mathscr{X}}, \bar{d})$ be its completion with an isometry $\phi \colon \mathscr{X} \to \bar{\mathscr{X}}$. We know that $\bar{\mathscr{X}} \times \bar{\mathscr{X}}$ is the completion of $\mathscr{X} \times \mathscr{X}$ via the isometry $\phi \times \phi$. We now need to put a vector space structure and an inner product on $\bar{\mathscr{X}}$ and verify that this inner product induces the metric on $\bar{\mathscr{X}}$. First the vector space structure.

We have an addition map $+ \colon \mathscr{X} \times \mathscr{X} \to \mathscr{X}$. This is Lipschitz because given $(a, b), (c, d) \in \mathscr{X} \times \mathscr{X}$

$$\|a + b - c - d\| \leq \|a - c\| + \|b - d\| \leq \sqrt{2}(\|a - c\|^2 + \|b - d\|^2)^{1/2}.$$

Therefore, addition is a Lipschitz map, hence extends to a map $\bar{+} \colon \bar{\mathscr{X}} \times \bar{\mathscr{X}} \to \bar{\mathscr{X}}$. The map $\bar{\mathscr{X}} \to \bar{\mathscr{X}}$ obtained by adding $\phi(0)$ is the extension of the identity map on $\mathscr{X}$, hence by uniqueness, $\phi(0)$ plays the role of additive identity.

Regarding the associativity of $\bar{+}$, consider the following commutative diagram.

$$\begin{array}{ccc} \mathscr{X} \times \mathscr{X} \times \mathscr{X} & \xrightarrow{+ \times id} & \mathscr{X} \times \mathscr{X} \\ {\scriptstyle id \times +} \downarrow & & \downarrow {\scriptstyle +} \\ \mathscr{X} \times \mathscr{X} & \xrightarrow[+]{} & \mathscr{X} \end{array}$$

Each of these maps can be lifted, and note that the lift of $+ \times id$ is $\bar{+} \times id$ by uniqueness. Since commutative diagrams lift to commutative diagrams, it follows that $\bar{+}$ is associative.

Similarly, one can extend the additive inverse map, and observe that the resulting lift is indeed the additive inverse (use the relation $x + (-x) = 0$). This way, $\bar{\mathscr{X}}$ is an additive group. One can similarly extend scalar multiplication, and verify that appropriate axioms hold (use corresponding commutative diagrams to check distributivity etc.), making it a vector space over F. Finally we need to extend the inner product.

Note that if $x_n \to x, y_n \to y$, then $x_n + y_n \to x \bar{+} y$ by the continuity of $\bar{+}$. Other similar results hold by the continuity of the vector space operations.

So far we were able to extend the vector space structure simply using the Lipschitz property of various maps. However, we cannot do the same for the inner product. This is because $\langle \cdot, \cdot \rangle$ grows quadratically (doubling both arguments results in a quadruple increase) while norm grows linearly. However, we have the following inequality for points $(x_1, y_1), (x_2, y_2) \in \mathscr{X} \times \mathscr{X}$

$$\begin{aligned} |\langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle| &\leq |\langle x_1 - x_2, y_1 \rangle + \langle x_2, y_1 - y_2 \rangle| \\ &\leq \|x_1 - x_2\| \, \|y_1\| + \|x_2\| \, \|y_1 - y_2\| \end{aligned}$$

In order to extend maps to the completion we needed the Lipschitz condition to hold so that we could ensure convergence of the image of a convergent sequence. Here however, we have something that is almost Lipschitz.

Given $(x, y) \in \bar{\mathscr{X}} \times \bar{\mathscr{X}}$, consider a sequence (x_n, y_n) converging to (x, y), then $x_n \to x, y_n \to y$ (to ease the notation we will assume that $\mathscr{X}$ is a subset of $\bar{\mathscr{X}}$). Since the sequence (x_n) is Cauchy, the norms are bounded, similarly $\|y_n\|$ is a bounded sequence, let M be a common bound. From the above inequality, $\langle x_n, y_n \rangle$ is a Cauchy sequence in F, hence convergent (because F is complete).

Next, suppose (x_n', y_n') is another sequence converging to (x, y), then applying the inequality above, it is clear that the sequences $\langle x_n, y_n \rangle, \langle x_n', y_n' \rangle$ converge to the same point in F. Therefore, we have an extension $\langle \cdot, \cdot \rangle_c$ of the inner product to $\bar{\mathscr{X}} \times \bar{\mathscr{X}}$. We next need to verify that this map is indeed an inner product and induces the corresponding metric.

Since we are taking the limit of sequences, it is easy to see that the extension $\langle \cdot, \cdot \rangle_c$ is indeed an inner product. As for the norm, note that given $x, y \in \bar{\mathscr{X}}$, suppose $x_n \to x, y_n \to y$. Then, $d(x_n, y_n) \to d(x, y)$, but at the same time, by definition of $\langle \cdot, \cdot \rangle_c$,

$$d(x_n, y_n) = \langle x_n - y_n, x_n - y_n \rangle^{1/2} \to \langle x - y, x - y \rangle_c^{1/2}$$

therefore $d(x, y) = \langle x - y, x - y \rangle_c^{1/2}$ as required. □

Remark. The same procedure applies to a normed linear space. In fact, it is simpler because we have

$$| \, \|x\| - \|y\| \, | \leq \|x - y\|$$

so the norm extends directly to its completion. Other properties are similarly verified. Note that to extend the vector space structure we only needed the norm on $\mathscr{X}$, so that part applies directly to normed spaces as to inner product spaces. In short, the completion of an inner product space is a Hilbert space and the completion of a normed linear space is a Banach space.

Complexification of a real hilbert space/banach space

References

[1] Conway, John B. *A course in Functional Analysis*