Baire Category Theorem and consequences

Shrivathsa Pandelu

April 10, 2021

1 Introduction, general remarks

Here I hope prove the Baire Category Theorem and some of its consequences, namely Open Mapping Theorem, Closed Graph Theorem, and the Bounded Inverse Theorem. None of this is original work and the proofs are taken from a number of places. Most of the content is from [1], [2] and [3] although I have also used [7].

2 Baire Category Theorem

Definition 1. *A topological space X is said to be a Baire space if for every countable collection $\{A_n\}_{n\geq 1}$ of closed sets with empty interior, their union $\cup_{n\geq 1}A_n$ also has empty interior in X. Equivalently, for every countable collection $\{U_n\}_{n\geq 1}$ of open dense sets, their intersection $\cap_{n\geq 1}U_n$ is also dense in X.*

Exercise Let A be a subset of a topological space X, then A has empty interior if and only if its complement is dense in X.

Example The space of rationals is not a Baire space because the singletons are closed with empty interior, but their union $\mathbb{Q}$ doesn't have empty interior. The space of integers, or any space with discrete topology is vacuosly Baire because there is no nonempty set with empty interior.

Remark. Originally R. Baire used the word "category". A subset A of a space X was said to be of the first category in X if it was contained in the union of a countable collection of closed sets of X having empty interiors; otherwise, it was said to be in the second category of X. Using this terminology, a space X is Baire if and only if every nonempty open set in X is in the second category.

The Baire category theorems are a collection of theorems which give sufficient conditions for a space to be a Baire space. It turns out that most reasonable spaces one deals with are Baire spaces.

Definition 2. *A pseudometric on a set X is a function $d\colon X\times X\to\mathbb{R}_{\geq 0}$ such that*

1. $d(x,x)=0\forall x\in X$.
2. $d(x,y)=d(y,x)\forall x,y\in X$.
3. $d(x,z)\leq d(x,y)+d(y,z)\forall x,y,z\in X$.

We can talk of pseudometric spaces, sequences in such spaces and their metric completions just the way one speaks of in metric spaces. The only difference between a pseudometric and a metric is that the "distance" between two different points under a pseudometric can be zero, i.e., the space need not be T_0.

Consider the space C of functions on $[0,1]$ which are square integrable. This space is equipped with a "norm"

$$\|f\|=\sqrt{\int_{[0,1]}f^2dt}.$$

This induces a pseudometric

$$d(f,g)=\sqrt{\int_{[0,1]}(f-g)^2dt}$$

which is not a metric because functions which are nonzero, but vanish almost everywhere have distance 0 from the zero function. In order to obtain a metric space structure, we quotient C by the subspace of functions which vanish everywhere to obtain the space $l^2([0,1])$.

Theorem 1. (*Baire Category Theorem 1*) *A complete pseudometric space is a Baire space.*

Proof. Let $\{A_n\}$ be a countable collection of closed sets with empty interiors. To show that $\cup A_n$ has empty interior, we must find, given any open set U, an element of U not contained in any of the A_ns. Let U be an open set.

Since A_1 has empty interior, there is a point $x_1 \in U \setminus A_1$, hence an $\epsilon_1 > 0$ such that $x_1 \in B(x_1, \epsilon_1) \subseteq U \setminus A_1$. Set $U_1 = B(x_1, \epsilon_1/4)$ and note that $\overline{U_1} \subset U \setminus A_1$.

Now, A_2 has empty interior, so there is a point $x_2 \in U_1 \setminus A_2$, hence an $\epsilon_2 > 0$ such that $x_2 \in B(x_2, \epsilon_2) \subseteq U_1 \setminus A_2$. Set $U_2 = B(x_2, \epsilon_2/4)$. Note that $x_2 \notin A_1 \cup A_2$ and that $\epsilon_2 < \epsilon_1/2$ and that $\overline{U_2} \subset U_1 \setminus A_2$.

Continuing this way, we obtain $x_n \in U_{n-1} \setminus A_n$ and hence an $\epsilon_n > 0$ such that $x_n \in B(x_n, \epsilon_n) \subseteq U_{n-1} \setminus A_n$, and we set $U_n = B(x_n, \epsilon_n/4)$. Then $x_n \notin A_1 \cup \cdots \cup A_n, \epsilon_n < \epsilon_1/2^{n-1}$ and $\overline{U_n} \subset U_{n-1} \setminus A_n$.

The diameter of U_n is less than $\epsilon_1/2^{n-1}$ and $\overline{U_n}$ contains the tail $x_n, x_{n+1}, \ldots$. The sequence $\{x_n\}_{n\geq 1}$ is Cauchy and hence converges to some x. Since $\overline{U_n}$ is closed, $x \in \overline{U_n} \subset U_{n-1} \setminus A_n$, hence $x \notin A_n$. Therefore, $x \in U$ is an element not in the union $\cup A_n$ as required, hence X is Baire. □

Theorem 2. (*Baire Category Theorem 2*) *A locally compact Hausdorff space is a Baire space.*

Proof. The proof is similar to the argument above and one uses finite intersection property instead of completeness. We know that if X is locally Hausdorff, and given a closed set A and $x \notin A$, there is a neighbourhood U of x such that $\overline{U}$ is compact and disjoint from A. Let $\{A_n\}$ be a countable collection of closed sets with empty interior and let U be any open set.

Since A_1 has empty interior, there is an $x_1 \in U \setminus A_1$, hence an open set U_1 such that $\overline{U_1} \cap A_1 = \emptyset, \overline{U_1} \subseteq U$ and $\overline{U_1}$ is compact.

Now, having obtained $\overline{U_{n-1}}$ we choose a point in $U_{n-1} \setminus A_n$ and then a neighbourhood U_n of this point so that

$$\overline{U_n} \cap A_n = \emptyset \text{ and } \overline{U_n} \subset U_{n-1}.$$

We then have a descending chain $\overline{U_1} \supset \overline{U_2} \supset \cdots \supset \overline{U_n} \supset \ldots$ and since $\overline{U_1}$ is compact, the intersection $\cap \overline{U_n}$ is nonempty. Let $x \in \cap \overline{U_n}$, then $x \notin \cup A_n$, hence X is a Baire space. □

Remark. In choosing the x_n or the U_n we had to invoke some form of the Axiom of Choice. In fact, the first theorem is equivalent over ZF to the Axiom of Dependent Choice. However, a restricted form of the theorem in which the complete metric space is also assumed to be separable is provable in ZF without choice. For more about this, see [4].

Lemma 1. *Any open subspace Y of a Baire space X is itself Baire.*

Proof. Suppose A is a closed subset of Y with empty interior. Then observe that $\overline{A}$ is a closed set in X with empty interior. For if U is an open set in X contained in $\overline{A}$, then $U \cap Y$ is open in Y and $U \cap Y \subseteq A$ which means that $U \cap Y = \emptyset$. Now however, if $z \in U \cap \overline{A}$, then U is a neighbourhood of $z \in \overline{A}$ but it doesn't intersect A which is a contradiction.

Let $\{A_n\}$ be a countable collection of closed sets in Y each with empty interior, then $\{\overline{A_n}\}$ is a countable collection of closed sets in X each with empty interior. Therefore, $\cup \overline{A_n}$ has empty interior in X. If U is an open subset of Y contained in $\cup A_n$, then it is open in X (for Y is an open subset) and is contained in $\cup A_n \subseteq \cup \overline{A_n}$ which means $U = \emptyset$. Therefore, $\cup A_n$ has empty interior in Y, hence Y is a Baire space. □

Theorem 3. *Let X be a space, (Y, d) a metric space. Let $f_n \colon X \to Y$ be a sequence of continuous functions converging pointwise to a function $f \colon X \to Y$. If X is a Baire space, the set of continuity points of f is dense in X.*

Proof. If f is continuous at a point x, then given any sequence x_n converging to x, we know that $f(x_n)$ converges to $f(x)$, i.e

$$\lim_n \lim_m f_m(x_n) = \lim_m f_m(x).$$

So, given a positive integer N and an $\epsilon > 0$, we set

$$A_N = \{x : d(f_n(x), f_m(x)) \leq \epsilon \text{ for all } n, m \geq N\}.$$

The set A_N is closed as it is the intersection over $m, n \geq N$ of the closed sets $\{x : d(f_n(x), f_m(x)) \leq \epsilon\}$ which are closed by the continuity of f_n, f_m.

For a given $\epsilon > 0$, we have $A_1(\epsilon) \subseteq A_2(\epsilon) \subseteq \ldots$ and their union is all of X by pointwise convergence of $f_n(x)$. Now let

$$U(\epsilon) = \cup_{N \geq 1} \mathrm{Int} A_N(\epsilon).$$

We shall show that

1. $U(\epsilon)$ is open and dense in X.
2. f is continuous at each point of $C = U(1) \cap U(1/2) \cap U(1/3) \cap \ldots$.

Given an open set V, note that the sets $V \cap A_N(\epsilon)$ are closed in V and their union is V, hence at least one of them must contain nonempty interior because V is a Baire space. This means that there is an open subset W of V contained in $V \cap A_M(\epsilon)$ for some M. Since V is open, W is open in X, hence W is contained in $\mathrm{Int} A_M(\epsilon)$, therefore $U(\epsilon)$ is dense.

Next, given $x_0 \in C$ and $\epsilon > 0$ choose k such that $1/k < \epsilon$. Since $x_0 \in U(1/k)$ there is a neighbourhood W such that $x_0 \in W \subseteq Int A_N(1/k)$ for some N. Since f_N is continuous, there is a neighbourhood $W_1 \subset W$ of x_0 such that $d(f_N(x), f_N(x_0)) < \epsilon$.

Now, since $W_1 \subset A_N(1/k)$, we have for each $n \geq N, x \in W_1$

$$d(f_n(x), f_N(x)) \leq 1/k < \epsilon$$

so, taking the limit over $n, d(f(x), f_N(x)) \leq 1/k < \epsilon \, \forall \, x \in W_1$. In particular, $d(f(x_0), f_N(x_0)) < \epsilon$.

Applying the triangle inequality, we obtain $d(f(x), f(x_0)) < 3\epsilon \, \forall \, x \in W_1$, therefore f is continuous at x_0 and this completes the proof. □

3 Applications to Functional Analysis

3.1 Open Mapping Theorem

Theorem 4. *Let X, Y be Banach spaces and $A \colon X \to Y$ be a continuous surjective linear map, then A is an open map.*

Proof. Suppose U is open in X and let $f(x) \in f(U)$. Consider the translate $U - x$ which is an open neighbourhood of 0 in X and its image, by linearity, is $f(U) - f(x)$ which contains the origin. If we show that $f(U) - f(x)$ contains a neighbourhood of $0 \in Y$, then by translating back, we know that $f(U)$ contains a neighbourhood of $f(x)$. Since f is surjective, this shows that $f(U)$ is an open set. Therefore, it suffices to consider open neighbourhoods of $0 \in X$, in particular it suffices to show that the image of open balls centred at the origin contains a neighbourhood around $0 \in Y$.

To this end, let $U = B_X(0,1), V = B_Y(0,1)$ be unit balls around the origin in X, Y respectively. By surjectivity, the closed sets $\{\overline{A(kU)}\}_{k \geq 1}$ cover Y, and by Baire Category Theorem, some $\overline{A(kU)}$ has nonempty interior.

Let us say $B_Y(c, r) \subset \overline{A(kU)}$ for some $c \in Y$ in which case, for every $v \in V, c + rv \in \overline{A(kU)}$. Using linearity, it follows that

$$rv \in \overline{A(kU)} + \overline{A(kU)} \subseteq \overline{A(2kU)}$$

where the containment comes from the following: any $x \in \overline{A(kU)}$ is the limit of a sequence in $A(kU)$.

Applying linearity once again, we obtain $V \subseteq \overline{A(lU)}$ where $l = 2k/r$. What this means is that given $v \in V$ and $\epsilon > 0$, there is an $x \in lU$, i.e., $x \in X$ with $\|x\|_X < l$ such that $\|v - Ax\| < \epsilon$.

We claim that $V \subseteq A(2lU)$, and then shrinking the sets would complete the proof. One idea is to take a sequence x_n such that $A(x_n)$ converge to v, and hope that the x_n converge to some x, and comparing the norms gives us the required containement, however there is no reason for the x_n to be a convergent sequence.

We instead approximate v as a sum. First observe that $V/n \subset \overline{A(l/nU)}$. First, obtain x_1 with $\|x_1\|_X < l$ such that

$$\|v - Ax_1\|_Y < 1/2.$$

Now, $v - Ax_1$ is an element closer to 0 than v. Obtain an $x_2 \in X$ with $\|x_2\|_X < l/2$ such that

$$\|v - Ax_1 - Ax_2\|_Y < 1/4.$$

Continuing this way, having obtained $x_1, \dots, x_{n-1}$, obtain an $x_n \in X$ with $\|x_n\|_X < l/2^{n-1}$ such that

$$\|v - A(x_1 + \dots + x_n)\|_Y < 1/2^n.$$

It is clear that $\sum x_n$ is defined in X, let it be x and by continuity of A, $Ax = v$. Moreover, $\|x\|_X < 2l$ as required. Therefore, $V \subseteq A(2lU)$. Therefore, $A(U)$ contains the ball $V/2l$. Hence A is an open map. □

If N denotes the kernel of A, then we have a vector space isomorphism $A\colon X/N \to Y$. Give X/N the quotient topology (via the quotient map $x \sim y \iff x - y \in N$), then this map is continuous. Since A is open, this resulting quotient map is also open, therefore, $A\colon X/N \to Y$ is a linear homeomorphism between vector spaces.

The theorem holds more generally under the following hypothesis, see[3].

1. X is an F-space, Y a topological vector space,
2. A is continuous and linear,
3. $A(X)$ is in the second category

in which case A is surjective, open and Y is an F-space. F-spaces are vector spaces over $\mathbb{R}$ or $\mathbb{C}$ with a complete metric which makes the vector space operations continuous, and is translation invariant. That Y is an F-space follows by passing to the quotient by the kernel of A and inducing an appropriate metric.

3.2 Closed Graph Theorem

Given a map $f\colon X \to Y$ between two topological spaces, the graph Γf is the subspace $\{(x, f(x)) : x \in X\} \subseteq X \times Y$. If f is continuous and Y Hausdorff, then it is easy to see that the graph Γf is a closed subset of $X \times Y$. We have two theorems that go by the name "Closed Graph Theorem" and we shall prove both.

Theorem 5. (*Topological*) *Let $f\colon X \to Y$ be a map between two spaces and let Y be compact Hausdorff, then f is continuous if and only if its graph is closed.*

Proof. If f is continuous, then it is clear that its graph is closed because Y is Hausdorff. Conversely, suppose Γf is closed in $X \times Y$. First we show that the projection $\pi\colon X \times Y \to X$ is a closed map. Let C be closed in $X \times Y$ and suppose $(x, y) \notin C$. Then for each $y \in Y$ there is a neighbourhood $U_y \times V_y$ containing (x, y) disjoint from C.

Using compactness of Y we obtain a finite subcover from the V_ys, and hence a neighbourhood U of X such that $U \times Y$ is disjoint from C, then U is disjoint from $\pi(C)$. Next, given a closed set $C \subseteq Y$ we know that $\Gamma f \cap X \times C$ is closed in $X \times Y$ and it's projection to X is precisely $f^{-1}(C)$ which is closed. This means that f is continuous. Note that we only need Y to be compact. □

Theorem 6. (*Functional analysis*) *Let $f\colon X \to Y$ be a linear map between two Banach spaces, then f is continuous if and only if it's graph is closed.*

Proof. Once again if f is continuous, then its graph is closed. Assume that the graph is closed. Now $X \times Y$ is a Banach space with the pythagorean (or any equivalent) norm say. Moreover, it is complete because a sequence is convergent/Cauchy iff its components are. Because f is linear and Γf is closed, it is also a Banach space with the restricted norm.

The projection $\pi_1 \colon \Gamma f \to X$ is a continuous, linear bijection, hence by the open mapping theorem it is an open map, i.e., $\pi_1^{-1} \colon X \to \Gamma f$ is a continuous map. Composing this with the projection $\pi_2 \colon X \times Y \to Y$ to the second coordinate, we see that $f = \pi_2 \circ \pi_1^{-1} \colon X \to Y$ is continuous as required. □

Remark. As in the case of the open mapping theorem, the closed graph theorem also holds more generally in the context of F-spaces. In fact, the proof above holds without change in the case of F-spaces.

3.3 Bounded Inverse Theorem

Theorem 7. *Let X, Y be a Banach space, $T \colon X \to Y$ a bounded linear map. If T is invertible, then T^{-1} is bounded.*

Proof. This is a direct consequence of the open mapping theorem. Since T is open, T^{-1} is continuous, hence bounded. □

Lemma 2. *A linear map $T \colon X \to Y$ is bounded if and only if it is continuous.*

Proof. Since we are dealing with metric spaces, continuity is equivalent to sequential continuity and by linearity it suffices to check continuity at $0 \in X$. If T is continuous at 0, then there is a $\delta > 0$ such that for $\|x\|_X < \delta$, $\|Tx\|_Y < 1$. Then given any $x \in X$, we have

$$\left\| T(\frac{\delta x}{\|x\|_X}) \right\|_Y = \frac{\delta}{\|x\|_X} \|Tx\|_Y \leq 1$$

hence $\|Tx\|_Y \leq (1/\delta) \|x\|_X$.

Conversely, suppose T is bounded, say $\|Tx\|_Y \leq C \|x\|_X$. Then given $\epsilon > 0$, for $\|x\|_X < \epsilon/C$ we have $\|Tx\|_Y < \epsilon$, hence T is continuous at 0. □

Remark. Although our proofs above requires proving the open mapping theorem first, the three theorems are actually equivalent (under reasonable assumptions), see [5] and [6].

3.4 Uniform Boundedness Principle (Banach-Steinhaus Theorem)

Theorem 8. *Let X be a Banach space, Y a normed linear space and suppose $\{T_n\}$ is a family of continuous linear transformations from X to Y such that for each $x \in X$, $\{\|T_n x\|\}$ is bounded, then $\|T_n\|$ is bounded.*

Proof. For each integer k, let $X_k = \{x : \sup_n \|T_n\| \leq k\}$, then each X_k is closed as if $x \notin X_k$, then $\|T_m(x)\| > k$ for some m which means that in some neighbourhood U of x, $\|T_m(y)\| > k \forall y \in U$, therefore $U \subseteq X_k^c$. By assumption, $\cup X_k = X$, so by Baire category theorem some X_k has nonempty interior, say X_N.

So, there is some $x_\in X, \epsilon > 0$ such that $\overline{B_X(x_0, \epsilon)} \subseteq X_N$. Now, given $u \in X$ with $\|u\| \leq 1$, we have for any $T = T_l$,

$$\begin{aligned}
\|Tu\| &= \epsilon^{-1} \|T(x_0 + \epsilon u) - T(x_0)\| \\
&\leq \epsilon^{-1} (\|T(x_0 + \epsilon u)\| + \|T(x_0\| \\
&\leq \epsilon^{-1} (N + N) = \epsilon^{-1}(2N)
\end{aligned}$$

Therefore, each T_l is bounded by the same constant. □

References and Further Reading

[1] Munkres, *Topology*

[2] John B. Conway, *A Course in Functional Analysis*

[3] Rudin, *Functional Analysis*

[4] Karagila, Asaf, Zornian Functional Analysis or: How I Learned to Stop Worrying and Love the Axiom of Choice

[5] R. S. Monahan, P. L. Robinson, The Closed Graph Theorem is the Open Mapping Theorem

[6] Henri Bourles, On the Closed Graph Theorem and the Open Mapping Theorem

[7] Wikipedia, Baire Category Theorem
Wikipedia, Open Mapping Theorem
Wikipedia, Bounded Inverse Theorem