Filtration, Inverse limits, Completion

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A compilation of a few proofs about filtrations, completions and inverse limits of rings and modules. These results are pretty standard and this write up is neither unique nor complete. Any book on Commutative Algebra is a good enough reference, see Commutative Algebgra by N. S. Gopakrishnan, Commutative Algebra by Matsumura, Commutative Algebra with a view towards Algebraic Geometry by Eisenbud for example.

1 Topological completion

Given a ring A, a filtration is a family of ideals $A = A_0 \supseteq A_1 \supseteq \dots$ such that $A_p A_q \subseteq A_{p+q}$. An A-module M is filtered if there are submodules $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ such that $A_p M_q \subseteq M_{p+q}$.
A topological group G is a group (similarly rings, modules etc.) with a topology such that the group operations are continuous. If G is such a group, then it suffices to know the neighbourhoods of the identity e as all other open sets are given by translations.

Proposition 1.1. *Let G be a topological group and let H be the intersection of all neighbourhoods of e.*

i) H is a subgroup.
ii) $H = \overline{\{0\}}$.
iii) G/H is Hausdorff.
iv) G is Hausdorff iff $H = 0$.

Proof. Left to the reader. □

Let M be a filtered A-module. Let τ be the topology generated by taking M_n as the neighbourhoods around 0. In this case, the M_n are also closed because the complement is given by $\cup_{x \notin M_n}(x + M_n)$. Similarly generate a topology on A. Addition is continuous because neighbourhood $x + M_n \times y + M_n$ maps to $(x+y) + M_n$. Multiplication is continuous as the neighbourhood $x + A_l \times m + M_n$ maps to $xm + M_n$ when $l \geq n$ (for $A_l m$ lands in M_{l+0}) (same argument for multiplication in A). With this kind of topology, A is a topological ring and M is a topological A-module.
The goal is to obtain a completion of M and we proceed in two ways which we prove to be equivalent. First, given a sequence (x_n) in M, we say it is Cauchy if for every k, there is an N such that for $m, n \geq N, x_n - x_m \in M_k$. Two Cauchy sequences $(x_n), (y_n)$ are equivalent if $(x_n - y_n)$ is Cauchy. It is easy to see that this is indeed an equivalence relation. Let $\tilde{M}$ denote the space of all equivalence classes of Cauchy sequences [1].
One easily sees that $\tilde{M}$ is an A-module and the obvious map $i: M \to \tilde{M}$ (constant sequences) is a module homomorphism. Moreover, it is injective iff M is Hausdorff. Set

$$\tilde{M_k} = \{[(x_n)] : \exists N \text{ such that } \forall n \geq N, x_n \in M_k\},$$

this is all those equivalence classes whose representatives, eventually, have elements from M_k. Note that the membership of any sequence to $\tilde{M_k}$ is independent of the representative chosen. It is also

[1] As far as set theory goes, I think this is a valid notion: form the space of sequences as all maps $\mathbb{N} \to M$, then form the subset of all Cauchy sequences using an axiom schema of specification which works because the property of being a Cauchy sequence is well defined and finally form a quotient as a subset of the power set under the map $x \mapsto [x]$ using axiom schema of replacement. Disclaimer: I am not completely sure about what I've written here.

clear that these $\tilde{M}_k$ are submodules and form a filtration of $\tilde{M}$ (verify that $A_l\tilde{M}_k \subseteq \tilde{M_{k+l}}$). Observe that an element in the intersection $\cap\tilde{M}_k$ is equivalent to the zero sequence. Moreover, the inclusion i is continuous with respect to the induced topology on $\tilde{M}$ because $i^{-1}(\tilde{M}_k) = M_k$ (technically, this would require $M_{k-1} \neq M_k$, but the argument works) with $\ker(i) = \cap M_k$.
Now we claim that $\tilde{M}$ is complete with the induced topology. Let (X_n) be a Cauchy sequence in $\tilde{M}$ and fix representatives $X_n = (x_{nm})$. For each i, there is a k_i such that for $r, s \geq k_i, X_r - X_s \in \tilde{M}_i$.
Let $m_1 \in \mathbb{N}$ be such that

$$x_{k_1t} - x_{k_2t} \in M_1, x_{k_2t} - x_{k_2s} \in M_2 \text{ for all } s, t \geq m_1.$$

Set $y_1 = x_{k_1m_1}$. Having found m_1, pick $m_2 > m_1$ such that

$$x_{k_2t} - x_{k_3t} \in M_2, x_{k_3t} - x_{k_3s} \in M_3 \text{ for all } s, t \geq m_2$$

and set $y_2 = x_{k_2m_2}$. Inductively construct the sequence $Y = (y_n)$. We claim that Y is a Cauchy sequence and that X_n converges to Y.
Observe that for $j \geq i$

$$y_i - y_j = y_i - y_{i+1} + y_{i+1} - y_{i+2} + \cdots + y_{j-1} - y_j \in M_i$$

because $y_i - y_{i+1} = x_{k_im_i} - x_{k_{i+1}m_i} + x_{k_{i+1}m_i} - x_{k_{i+1}m_{i+1}} \in M_i$ by definition of m_i and because $M_i \supseteq M_{i+1}$. For this reason, Y is a Cauchy sequence.
Given i, let t be such that $m_t > m_i$. Now, for $r \geq k_i$, we have

$$x_{rt} - y_t = x_{rt} - x_{k_tm_t} = x_{rt} - x_{k_it} + x_{k_it} - x_{k_im_i} + x_{k_im_i} - x_{k_tm_t} \in M_i.$$

Therefore, $X_r - Y \in \tilde{M}_i$ for $r \geq k_i$ because $m_{t+j} > m_t > m_i$ for any j. Thus, $\tilde{M}$ is complete. Since $\cap\tilde{M}_k = \{0\}$, we see that $\tilde{M}$ is a Hausdorff complete space.

Theorem 1.1. *(Universality) Let $\tilde{M}$ be as above and suppose Y is a filtered A-module*[2] *which is complete and Hausdorff. Given a continuous A-module homomorphism $f: M \to Y$, there is a unique extension (continuous, A-module homomorphism) $\tilde{f}: \tilde{M} \to Y$ such that $\tilde{f} \circ i = f$.*

Proof. Let $Y = Y_0 \supseteq Y_1 \supseteq \ldots$ be the filtration of Y. Given a sequence (x_n), the sequence $f(x_n)$ is Cauchy in Y because the inverse of any Y_k contains some M_l by continuity.
Since Y is Hausdorff, let there is a unique limit $\tilde{f}((x_n))$. This defines $\tilde{f}$. Using continuity of operations on Y (verify), $\tilde{f}$ is an A-module homomorphism. Let $U = y + Y_k$ be fixed and say $\tilde{f}(X) \in y + Y_k$ for some $X = (x_n)$. Since f is continuous, there is some t such that $f(M_t) \subseteq Y_k$. By convergence, there is an N such that $f(x_n) \in y + Y_k$ for $n \geq N$.
Now, given $(b_n) \in \tilde{M}_t$, there is a $P \geq N$ such that for $r \geq P, b_r \in M_t$ so $f(x_r + b_r) = f(x_r) + f(b_r) \in y + Y_k$. Therefore, $\tilde{f}(X + (b_n)) \in y + Y_k$ and hence $X + \tilde{M}_t \subseteq \tilde{f}^{-1}(y + Y_k)$ and we have continuity. □

2 Inverse limit completion

There is an algebraic way to obtain a completion. In general, suppose we have a poset I and a contravariant functor $\mathsf{A}\colon I \to \mathcal{M}_A$ where $\mathcal{M}_A$ is our temporary notation for the category of A-modules. Set $N_i = \mathsf{A}(i), i \in I$ and $\alpha^{ji}\colon N_j \to N_i$ to be $\mathsf{A}(i \leq j)$. The N_i are said to form an inverse system (some authors require I to be a directed poset, but we will not need that here (partly because we will be dealing with the directed/totally ordered set $\mathbb{N}$)). The inverse limit, if it exists, is an object $\hat{N}$ (the standard notation is: $\varprojlim N$) with maps $\alpha_i\colon \hat{N} \to N_i$ such that

$$\begin{tikzcd} & \hat{N} \arrow[dl, "\alpha_i"'] \arrow[d, "\alpha_j"] \\ N_i & N_j \arrow[l, "\alpha^{ij}"] \end{tikzcd}$$

[2]The author would have liked a more general result with Y being any complete T_2 space, but was limited by category theory.

commutes and it is universal with respect to this property (i.e., if there is another P satisfying this, then there is a unique map $P \to \hat{N}$ making all diagrams commute; universality gives uniqueness).
As far as existence goes, consider the set $\hat{N}$ of all coherent sequences, i.e., those sequences $(x_i)_{i \in I}$ such that $x_i \in N_i$ and $x_i = \alpha^{ji}(x_j)$ for $i \leq j$. This can be seen as a subset of the product of N_i (which exists because $\mathcal{M}_A$ has products). $\hat{N}$ has the obvious A module structure, and there are the projection maps $\hat{N} \to N_i$ (obtained by restricting the projection on the product; again this is allowed in $\mathcal{M}_A$). The universal property is easily verified.
Now, suppose $I = \mathbb{N}$ and we have a system $N_1 \leftarrow N_2 \leftarrow N_3 \leftarrow \dots$. Let $\hat{N}$ be the inverse limit, then we have the kernels $\hat{N_k} = \ker(\hat{N} \to N_k)$ and these form a filtration of $\hat{N}$ (only a filtration of an abelian group, this filtration doesn't necessarily behave nicely with a filtration on A, although we aren't a priori assuming a filtration on A yet). It is clear that the intersection is zero, therefore, with the induced topology, $\hat{N}$ is Hausdorff.
If X_n is a Cauchy sequence in $\hat{N}$, then for any k there is an integer n_0 such that for $r, s \geq n_0, X_r - X_s \in \hat{N_k}$ which means that the first k elements of X_r, X_s are the same. In other words, the coordinates of X_n are eventually constant and form a sequence Y. It is clear that X_n converge to Y, therefore $\hat{N}$ is a complete Hausdorff space.
Before we continue, we make the following observation. We defined our inverse limit as a subset of the product ΠN_i. Now, it is possible that each N_i has a topology (such that $N_{i-1} \xleftarrow{N}_i$ is continuous and N_i are topological groups/modules), in which case the product, hence the inverse limit, has a product topology. How does the topology induced from the above filtration compare to the subspace topology? Now, it is not difficult to see that in the event N_i are topological groups, then the product ΠN_i is also a topological group. Moreover, the inclusion $\hat{N} \to \Pi N_i$ is a group homomorphism and therefore we only need to look at things near 0.
Continuity is clear because the basic neighbourhoods of 0 on the left side are of the form $U_1 \times U_2 \times \dots \times U_k \times N_{k+1} \times \dots$ whose inverse would contain $N_k\hat{+}1$ for example. However, not that the basic neighbourhoods around 0 on the left would contain coherent sequences whose first few terms are all 0. Therefore, if each N_i was given the discrete topology, then the topology on $\hat{N}$ as a subspace of ΠN_i and that induced by filtration would agree.
Furthermore, because $\hat{N}$ is complete, it is easy to see that it is closed as a subset of ΠN_i (basically take neighbourhoods around any point in the closure to construct a Cauchy sequence in $\hat{N}$ convering to that point). Therefore, if N_i are finite, then $\hat{N}$ is compact.

2.1 Equivalence

Let M be an A-module with filtration $M = M_0 \supseteq M_1 \supseteq \dots$. We form an inverse system:

$$M_0/M_1 \leftarrow M_0/M_2 \leftarrow \dots$$

and let $\hat{M}$ denote the inverse limit. Its elements are equivalence classes $([x_0], [x_1], \dots)$ with $[x_i] \in M_0/M_{i+1}$ such that $x_{i+1} - x_i \in M_i$ (for any choice of representatives of the equivalence classes).
The filtration on $\hat{M}$ is given by $\hat{M}_k = \ker(\hat{M} \to M_0/M_{k+1})$. Because the filtration on M behaves nicely with the filtration on A, we observe that this filtration on $\hat{M}$ also behaves nicely with that on A, i.e., $\hat{M}$ is a filtered A-module.
Next, we also have the map

$$\begin{aligned} j\colon M &\to \hat{M} \\ x &\mapsto ([x], [x], \dots) \end{aligned}$$

and this is continuous because $j^{-1}(\hat{M}_k) = M_k$ and $\ker(j) = \cap M_k$.
We now have two spaces, $\hat{M}, \tilde{M}$ both of which are Hausdorff complete spaces. Now, given a Cauchy sequence $(x_n) \in \tilde{M}$, for any k, we have the quotient sequence in M/M_k. Since the sequence is constant, the quotient sequence is eventually constant, say some equivalence class ζ_k. Consider the map

$$\begin{aligned} F\colon \tilde{M} &\to \hat{M} \\ (x_n) &\mapsto (\zeta_n) \end{aligned}$$

Because we have Cauchy sequences, it is easy to see that the map is well defined (one needs to verify that the image is independent of the choice of the representative and that it lands in $\hat{M}$, both are easy checks) and an A-module homomorphism. It is injective as if a sequence (x_n) maps to zero, then it means that the tails are in M_k for every k, hence the sequence is equivalent to the zero sequence. F is also surjective as given an element $([y_1],[y_2],\dots) \in \hat{M}$, let $(y_1, y_2, \dots)$ be the sequence of coset representatives. This is a Cauchy sequence because $y_{k+i} - y_k \in M_k$ and immediately we also see that $F((y_n)) = ([y_n])$.
Thus, F is an A-module isomorphism. Observe that $F(\tilde{M}_k) = \hat{M}_k$, therefore F is a continuous and open map, i.e., F is a homeomorphism.
Note that the following diagram commutes

$$\begin{tikzcd} \tilde{M} \arrow[rr, "F"] & & \hat{M} \\ & M \arrow[ul, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

and M is dense in $\tilde{M}$: given an open set $(x_n) + \tilde{M}_k$, there is an N such that $x_n - x_N \in M_k$ for $n \geq N$, so the constant sequence $i(x_N)$ is in $(x_n) + \tilde{M}_k$.
While the inverse limit construction is more algebraic, it still carries some geometric intuition. One can imagine it as consisting a sequence that zeroes in on topological "holes", i.e., those points that the completion adds. Put simply, its elements are addresses: the element $([y_1],[y_2],\dots)$ corresponds to that element which is in the y_1 coset of M_1, y_2 coset of M_2 and so on. I imagine partitioning M first into translates of M_1, then each of those partitions into translates of M_2 and so on, creating a grid like partitioning of M and the inverse limit as a locating system that gives off the coordinates of the partitions.

3 Exactness

Suppose we have a filtered ring A and a filtered A-module M, with filtration $M = M_0 \supseteq M_1 \supseteq \dots$. We look at submodules and quotients first. Let N be a submodule, then there is the induced filtration $N_k = N \cap M_k$ on N and on $P = M/N$ we have the filtration $P_k = (M_k + N)/N \cong M_k/(M_k \cap N)$ and these give completions $\hat{N}, \hat{P}$.

- <u>The subspace topology agrees with the filtered topology:</u> it is clear that the filtered topology is contained in the subspace topology because for $n \in N, n + N \cap M_k = N \cap (n + M_k)$. Conversely, let $(x + M_k) \cap N \neq \emptyset$, say $n = x + m$ for $m \in M_k, n \in N$. Then it is easy to see that (M_k is a submodule)

$$(x + M_k) \cap N = n + (N \cap M_k).$$

  So, the inclusion $N \hookrightarrow M$ is continuous.
- <u>The quotient topology agrees with the filtered topology:</u> Let $q\colon M \to P$ denote the quotient map. If $x \in q^{-1}([y]+P_k)$, then $x+N \in y+(M_k+N)/N \implies x \in y+M_k+N$. And conversely, if $x \in y+M_k+N$, then it is clear that $q(x) \in [y]+P_k$. Therefore, $q^{-1}([y]+P_k) = y+M_k+N = \cup_{y+n\in N}(n+M_k)$ is an open set. Hence each $[y]+P_k$ is open, i.e., the filtered topology is contained in the quotient topology. Let U be open in the quotient topology, then $q^{-1}(U)$ is open. Given $[y] \in U$, there is some k such that $y + M_k \subseteq q^{-1}(U)$, which means that $[y] + P_k \subseteq U$, hence the quotient topology is equal to the filtered topology. So, the projection q is continuous.
- Because the completion is Hausdorff, we directly have the following extensions

$$\begin{tikzcd} \hat{N} \arrow[r, dashed] & \hat{M} \arrow[r, dashed] & \hat{P} \\ N \arrow[u] \arrow[r, hook] & M \arrow[u] \arrow[r, two heads] & P \arrow[u] \end{tikzcd}$$

Now comes the natural question of exactness, so we must deal with the category $\mathcal{F}$ of filtered A-modules, whose objects are filtered A-modules and morphisms are module homomorphisms which respect the filtration. Above, for example, the inclusion and quotient respect the filtration. Then there is the category $\mathcal{I}$ of inverse systems of A modules with respect to a fixed poset I (in our case, $\mathbb{N}$) whose objects are inverse systems and morphisms are those that make all diagrams in sight commute.
There is the functor

$$\mathsf{F} \colon \mathcal{F} \to \mathcal{I}$$

which sends a filtration $M = M_0 \supseteq M_1 \supseteq \dots$ to the inverse system $M/M_1 \longleftarrow M/M_2 \longleftarrow \dots$ and sending morphisms in the obvious manner. It is easy to see that this is a covariant functor.
Exactness in $\mathcal{F}, \mathcal{I}$ is one that is exact at each ”level”. We ask whether F is exact. Suppose we have the exact sequence $0 \to N \xrightarrow{f} M \xrightarrow{g} P \to 0$ of filtered A-modules:

$$\begin{tikzcd}
0 \arrow[r] & N_0 \arrow[r, "f"] & M_0 \arrow[r, "g"] & P_0 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & N_1 \arrow[r, "f"] \arrow[u, "i_1"] & M_1 \arrow[r, "g"] \arrow[u, "j_1"] & P_1 \arrow[r] \arrow[u, "k_1"] & 0 \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

Above we have a slightly more general notion of a filtration where i_l, j_l, k_l are some **injective** morphisms and all diagrams commute (so, we have an inverse system). Another way is to instead take this to be the definition of a filtration, but as far as this note is concerned, it suffices to take these to be inclusions[3]. We can take appropriate quotients and we have the following maps:

$$\begin{tikzcd}
0 \arrow[r] & N_0/i_1(N_1) \arrow[r, "f"] & M_0/j_1(M_1) \arrow[r, "g"] & P_0/k_1(P_1) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & N_0/i_1(i_2(N_2)) \arrow[r, "f"] \arrow[u, "i_1"] & M_0/j_1(j_2(M_2)) \arrow[r, "g"] \arrow[u, "j_1"] & P_0/k_1(k_2(P_2)) \arrow[r] \arrow[u, "k_1"] & 0 \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

We claim that this sequence is exact. We will prove for the first row, but the rest are similar.
First, suppose $f(n + i_1(N_1)) = f(n) + j_1(M_1) = 0$, so $f(n) = j_1(m)$ for some $m \in M_1$. Then $k_1(g(m)) = g(j_1(m)) = g(f(n)) = 0$, hence $g(m) = 0$, so there exists an $n' \in N_1$ such that $f(n') = m$. Then

$$f(i_1(n')) = j_1(f(n')) = j_1(m) = f(n)$$

hence, $i_1(n') = n$, i.e, $n + i_1(N_1) = 0 \in N_0/N_1$.
It is clear that we have a complex at M_0/M_1, i.e., $g \circ f = 0$. Suppose $g(m + j_1(M_1)) = 0$, then $g(m) \in k_1(P_1)$. By surjectivity of $g \colon M_1 \to P_1$, there is an $m' \in M_1$ such that $g(m) = k_1(g(m'))$. Then

$$g(m) = g(j_1(m')) \implies m - j_1(m') = f(n) \text{ for some } n \in N_0$$

Therefore, $m + j_1(M_1) = f(n + i_1(N_1))$.
Finally, surjectivity is clear because $M_0 \to P_0 \to P_0/k_1(P_1)$ is surjective.
Therefore, F is an exact functor.

[3] Because these are injective, it is possible to instead take an isomorphic filtration where all maps are in fact inclusions.

Next, we have another functor

$$\mathsf{C}\colon \mathcal{I} \to \mathcal{M}_A$$

which sends an inverse system to its inverse limit, which is unique by its universal property and this is indeed a functor by the universal property. Let's say we have an exact sequence of inverse systems as follows[4]:

$$\begin{tikzcd}
0 \arrow[r] & N_0 \arrow[r, "f"] & M_0 \arrow[r, "g"] & P_0 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & N_1 \arrow[r, "f"] \arrow[u, "i_1"] & M_1 \arrow[r, "g"] \arrow[u, "j_1"] & P_1 \arrow[r] \arrow[u, "k_1"] & 0 \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}$$

We then have the induced maps

$$0 \to \varprojlim N \xrightarrow{\tilde{f}} \varprojlim M \xrightarrow{\tilde{g}} \varprojlim P \to 0$$

If a sequence $(x_n) \in \varprojlim N$ maps to zero, then it is clear by exactness at N_n that each $x_n = 0$. It is also clear that $\tilde{g} \circ \tilde{f} = 0$. Now, suppose a sequence $(m_n) \in \varprojlim M$ maps to 0 under $\tilde{g}$, then $g(m_0) = 0$, hence there is some $n_0 \in N_0$ such that $f(n_0) = m_0$. Next, $g(m_1) = 0$, so we can find $n_1 \in N_1$ such that $f(n_1) = m_1$. Is $i_1(n_1) = n_0$? This is forced because f is injective and $f(n_0) = m_0 = f(i_1(n_1))$. It thus follows that there exists a seqeunce in $\varprojlim N$ mapping to (m_n). Thus, we have the exact sequence

$$0 \to \varprojlim N \xrightarrow{\tilde{f}} \varprojlim M \xrightarrow{\tilde{g}} \varprojlim P$$

Given a sequence $(p_n) \in \varprojlim P$ first pick $m_0 \in M_0$ such that $g(m_0) = p_0$. Next, we can find a m_1 with $g(m_1) = p_1$, but the question is whether $j_1(m_1) = m_0$. We have

$$g(j_1(m_1)) = k_1(g(m_1)) = k_1(p_1) = p_0 = g(m_0)$$

hence $j_1(m_1) - m_0 = f(n_0)$ for some $n_0 \in N_0$. If i_1 is **surjective**, then we can find an $n_1 \in N_1$ such that $n_0 = i_1(n_1)$ and then

$$j_1(m_1 - f(n_1)) = m_0 \text{ and } g(m_1 - f(n_1)) = g(m_1) = p_1.$$

In this way we can construct a sequence (m_n) which maps to (p_n) under $\tilde{g}$ with the requirement that all i_l be **surjective**.
In the case of an exact sequence of filtered modules, we see that we get an exact sequence of completions.

4 Graded rings

A ring R is said to be graded if it can be written as a direct sum of abelian groups:

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \dots$$

such that $R_i R_j \subseteq R_{i+j}$. This implies that R_0 is a subring and $R_i, i \geq 1$ are R_0 modules. The ideal $\oplus_{i \geq 1} R_i$ is called the irrelevant ideal. From a graded ring we have the filtration

$$R \supseteq \oplus_{i \geq 1} R_i \supseteq \oplus_{i \geq 2} R_i \supseteq \dots$$

and from there we can talk about the completion of R.

[4]For those wondering, yes, this is the exact diagram as before.

Typical examples of graded rings are polynomial rings with the gradation determined by degree. Similarly we define a graded R-module $M = \oplus_{i\geq 0} M_i$. The elements of R_i, M_i are called homogeneous elements.
If $R = R_0 \supseteq R_1 \supseteq \dots$ is a filtered ring, then we set $gr_n(R) = R_n/R_{n+1}$ and $gr(R) = \oplus_{n\geq 0} gr_n(R)$ (an external direct sum) with multiplication

$$(a + R_{n+1})(b + R_{m+1}) = ab + R_{m+n+1}, a \in R_n, b \in R_m.$$

$gr(R)$ is called the associated graded ring of R. Similarly we can define the associated graded module $gr(M)$ of a filtered R-module M and this has a $gr(R)$-module structure.
Thus we can go from the category of filtered R-modules to the category of graded R-modules and it is easy to see that this is a functor.

Theorem 4.1. *Let $R = \oplus_{n\geq 0} R_n$ be a graded ring. R is Noetherian iff R_0 is Noetherian and R is a finitely generated R_0 algebra.*

Proof. Suppose R is Noetherian. Then R_0 being a quotient of R is Noetherian. Let I denote the irrelevant ideal generated by some $x_1, \dots, x_n$ of degree $d_1, \dots, d_n$ respectively over R. Given an element $x \in R_N$ say, write it as an R-linear combination of x_is. Now the coefficients themselves are graded and because x has degree N, all homogenous terms of higher degree have to cancel out. In this way, we reduce the degree of the coefficient of x_i to be less than $N - d_i$.
We repeat this for the coefficients themselves and in a finite number of steps arrive at a polynomial expression in x_is with coefficients from R_0. Therefore, R is a finitely generated R_0-algebra.
Conversely, when R_0 is Noetherian, by Hilbert basis theorem, R, a finitely generated R_0-algebra, is Noetherian. □

Fix an ideal I of R and consider the I-adic filtration $R = I^0 \supseteq I \supseteq I^2 \supseteq \dots$. Let $\{M_n\}$ be a filtration on an R-module M. The filtration is said to be an I-**filtration** if $IM_n \subseteq M_{n+1}$ and it is I-**stable** if $IM_n = M_{n+1}$ for large n. Temporarily denote by R^* the ring $\oplus_{n\geq 0} I^n$ and $M^* = \oplus_{n\geq 0} M_n$, the graded R^*-module.

Proposition 4.1. *Given an ideal I of a ring A, any two I-stable filtrations on an A-module M give isomorphic completitions.*

Proof. Let $M = M_0 \supset M_1 \supset \dots$ and $M = N_0 \supset N_1 \supset \dots$ be the two I-stable filtrations. The main tool we use is the universality theorem because the completitions are complete Hausdorff spaces. We may without loss of generality assume that $N_i = I^i M$. All we need to show is that the identity map is continuous. We have for every $d, M_{s+d} = I^d M_s \subseteq I^d M$ for some large s and we also for every $d, I^d M \subseteq M_d$. So, the identity maps on M with these filtrations is continuous giving us a homeomorphism between the completitions (homeomorphism follows from the uniqueness). □

Lemma 4.1. *(Artin-Rees lemma lemma) For a finitely generated R-module M over a Noetherian ring R, an I-filtration $\{M_n\}$ is I-stable iff M^* is a finitely generated R^* module.*

Proof. Suppose the filtration is I-stable, say $IM_n = M_{n+1}$ for $n \geq N$. The submodules $M_0, \dots, M_N$ are finitely generated over R. By the I-stable property, every other element of M^* is seen as an R^*-linear combination of the finitely many generators of $M_0 \oplus \dots \oplus M_N$.
Conversely, suppose M^* is finitely generated over R^* and these generators together lie in $\oplus_{i=0}^N M_i$. We may assume that the generators are homogeneous, i.e., lying in some M_k. Given $x \in M_n, n > N$ we write it as an R^* linear combination of these generators. By comparing the degrees we conclude that the degree of the coefficient of a generator in M_k must be $n - k$. Because it's an I-filtration, this product can be seen as an element of IM_{n-1}. Because $n > N$, we can't have $k = n$. Thus, we conclude that $M_n \subseteq IM_{n-1}$. The other inclusion holds by the I-filtration hypothesis. □

Theorem 4.2. *(Artin-Rees lemma) Let R be a Noetherian ring, I an ideal of R and N a submodule of a finitely generated R-module M. Then any filtration on N induced by an I-stable filtration on M is I-stable.*

Proof. Let $M = M_0 \supseteq M_1 \supseteq \dots$ be the filtration on M and $N = N_0 \supseteq N_1 \supseteq \dots$ be the induced filtration on N and let R^*, M^*, N^* be as above. We know R^* is Noetherian because R is Noetherian and R^* is generated as an algebra over R by the generators of I.
Because M is Noetherian (being finitely generated over R) and the filtration is I-stable, M^* is finitely generated over R^*, hence is Noetherian, which means the submodule N^* is finitely generated over R^* as a module.
Thus, N^* is a finitely generated R^*-module, which means that the induced filtration is I-stable. □

Because the I-adic filtration on M is I-stable we have the following

Corollary 4.1. *With M, N as above there exists $n_0 > 0$ such that for every $n \geq n_0$*

$$I^n M \cap N = I^{n-n_0}(I^{n_0} M \cap N).$$

Theorem 4.3. *(Krull's intersection theorem) Let R be a Noetherian ring and M a finitely generated R-module. Let $N = \cap_{n\geq 1} I^n M$ then there is a $b \in R$ such that $(1+b)N = 0$.*

Proof. This is a simple application of Artin-Rees and Nakayama lemma. There is an m such that for $n \geq m$,

$$N = I^n M \cap N = I^{n-m}(I^m M \cap N) = I^{n-m} N$$

thus, there is a $b \in I^{n-m}$ such that $(1+b)N = 0$. □

Above, if $I \subseteq \mathfrak{R}(R)$, is contained in the Jacobson radical, then $N = 0$ because $1 + b$ would be a unit. In particular, $N = 0$ in any local ring. Topologically, Krull intersection theorem says that the I-adic filtration is Hausdorff when I is contained in the Jacobson radical. Moreover, for any submodule M_1 we have

$$\overline{M_1}/M_1 = (\cap(M_1 + I^n M))/M_1 \subseteq \cap(M_1 + I^n M)/M_1 = \cap I^n(M/M_1) = 0$$

so, submodules are closed in the I-adic topology.

Proposition 4.2. *Let $\mathfrak{m}_1, \dots, \mathfrak{m}_k$ be distinct maximal ideals of a Noetherian ring R and let $I = \cap \mathfrak{m}_i$. Let $\hat{R}_{\mathfrak{m}_i}$ denote the $\mathfrak{m}_i$-adic completion of R and $\hat{R}$ the I-adic completion of R, then $\hat{R} \cong \Pi \hat{R}_{\mathfrak{m}_i}$.*

Proof. By the Chinese remainder theorem, we know that $R/I^n = \Pi R/\mathfrak{m}_i^n$ and it is an easy thing to verify that inverse limit distributes over products, i.e., if $A_1 \leftarrow A_2 \leftarrow \dots, B_1 \leftarrow B_2 \leftarrow \dots$ are two inverse systems with inverse limits $\hat{A}, \hat{B}$, then $\hat{A} \times \hat{B} \cong \widehat{A \times B}$ via the natural map where $\widehat{A \times B}$ is the inverse limit of $A_1 \times B_1 \leftarrow A_2 \times B_2 \leftarrow \dots$. Thus, $\hat{R}$ is isomorphic to $\Pi \hat{R}_{\mathfrak{m}_i}$ as rings via the map

$$(x_i) \mapsto \Pi(x_i).$$

obtained from the isomorphism $R/I^n \to \Pi R/\mathfrak{m}_i^n$ which is just the projection map.
Because of this isomorphism, it's not hard to see that the image of $\ker(\hat{R} \to R/I^n)$ is precisely $\Pi \ker(\hat{R}_{\mathfrak{m}_i} \to R/\mathfrak{m}_i^n)$, thus this map is an open map (we have open-ness around zero and therefore open-ness everywhere). Sets of the latter type are contained inside any basic neighbourhood around $0 \in \Pi \hat{R}_{\mathfrak{m}_i}$, hence we have continuity at zero, therefore, continuity everywhere and this completes the proof. □

Proposition 4.3. *Let R be a ring, I an ideal. If R is I-adic complete, then $I \subseteq \mathfrak{R}(R)$.*

Proof. R being I-adic complete means that the map $R \to \hat{R}$ is an isomorphism. To show that I is contained in the Jacobson radical, it suffices to show that $1 - x$ is a unit for every $x \in I$. Given an $x \in I$, the formal inverse of $1 - x$ would be given by $1 + x + x^2 + \dots$. Observe that the partial sums form a Cauchy sequence in the I-adic topology, hence converge to some $y \in R$ and because multiplication is continuous and $x^n \to 0$ in the I-adic topology, we conclude that $(1-x)y = 1$. □

5 Examples

The p-adic integers $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to the ideal (p). From an earlier observation, $\mathbb{Z}_p$ is a compact complete ring. Another important example is the power series ring $k[[x_1,\dots,x_n]]$ which is the I-adic completion of $k[x_1,\dots,x_n]$.

Theorem 5.1. *(Hansel's lemma) Suppose $(A,\mathfrak{m})$ is a complete local ring and $F \in A[x]$ is a monic polynomial with reduction $f \in k[x]$ where $k = A/\mathfrak{m}$ is the residue field of A. Suppose $f(x) = g(x)h(x)$ in $k[x]$ for coprime monic polynomials g,h. Then there exist monic polynomials $G,H \in A[x]$ such that $F(x) = G(x)H(x)$ and such that G,H reduce to g,h modulo $\mathfrak{m}$ respectively.*

Proof. We inductively factorse F modulo $\mathfrak{m}^n$. For $n = 1$, we have

$$F = G_1 H_1 \mod \mathfrak{m}$$

where G_1,H_1 are polynomials over A reducing to g,h respectively. Assuming we have found G_n,H_n such that $F - G_nH_n = 0 \mod \mathfrak{m}^n$ and G_n,H_n reduce to g,h modulo $\mathfrak{m}$ we write

$$F - G_nH_n = \sum P_iQ_i$$

where $P_i \in \mathfrak{m}^n$ and $\deg Q_i < \deg F$. Since g,h are relatively prime, going modulo $\mathfrak{m}$ and then lifting to A, we can write

$$Q_i = GR_i + HS_i + b_i$$

where b_i are polynomials over $\mathfrak{m}$.
Set $G_{n+1} = G_n + \sum P_iR_i, H_{n+1} = H_n + \sum P_iS_i$, then

$$G_{n+1}H_{n+1} = G_nH_n + \sum P_i(G_nR_i + H_nS_i)$$

Modulo $\mathfrak{m}$, the last term becomes $P_iQ_i + e$ where e is the extra terms which are in $\mathfrak{m}^{n+1}$. Therefore $F \equiv G_{n+1}H_{n+1} \mod \mathfrak{m}^{n+1}$. It is also clear that G_{n+1},H_{n+1} reduce to g,h modulo $\mathfrak{m}$.
Inductively we obtain G_n,H_n. In our construction, $G_{n+1} - G_n, H_{n+1} - H_n \in \mathfrak{m}^n$ so these are Cauchy sequences, hence converge to some $G,H \in A[x]$. It then follows that $F = GH$ because the difference is in $\mathfrak{m}^n$ for every n and because we have a local ring, Krull's intersection theorem tells that A is Hausdorff, so $F - GH$ has to be zero. □

Hansel's lemma gives us a way to construct solutions to polynomials over $\mathbb{Z}$ in $\mathbb{Z}_p$.